Topological methods in analysis of periodic and chaotic canard-type trajectories

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1 Introduction

This paper investigates the role of topological methods in the analysis of canard-type periodic and chaotic trajectories. In Sections 1 − 5 we apply topological degree [1, 2] to the analysis of multi-dimensional canards. This part of the paper was written mainly by the first and the last authors. Sections 6 − 7 are devoted to an application of a special corollary of the Poincaré–Bendixson theorem to the existence of periodic two-dimensional canards. This fragment was written mainly by the first and the second authors.

If $W\colon \mathbb{R}^d \to \mathbb{R}^d$ is a continuous mapping, $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $y \in \mathbb{R}^d$ does not belong to the image $W(\partial\Omega)$ of the boundary $\partial\Omega$ of Ω, then the symbol $\deg(W, \Omega, y)$ denotes the *topological degree* [1] of W at y with respect to Ω. If $0 \notin W(\partial\Omega)$, then the integer number $\gamma(W, \Omega) = \deg(W, \Omega, 0)$, called the *rotation of the vector field W at $\partial\Omega$*, is well defined. A detailed description of properties of the number $\gamma(W, \Omega)$ can be found, for example, in [2]. In particular, if I denotes the identity mapping, $I(x) \equiv x$, then the number $\gamma(I - W, \Omega)$ measures the algebraic number of fixed points of the mapping W in Ω.

Consider the slow-fast system

$$\begin{aligned} \dot{x} &= X(x, y, \varepsilon) + \hat{X}(x, y, z, \varepsilon),\\ \varepsilon\dot{y} &= Y(x, y, \varepsilon) + \hat{Y}(x, y, z, \varepsilon),\\ \dot{z} &= Z(x, y, z, \varepsilon). \end{aligned} \tag{1}$$

Here

$$x \in \mathbb{R}^2, \quad y \in \mathbb{R}^1, \quad z \in \mathbb{R}^d,$$

and $\varepsilon > 0$ is a small parameter. The terms $\hat{X}(x, y, z, \varepsilon)$ and $\hat{Y}(x, y, z, \varepsilon)$ are supposed to be small with respect to the uniform norm:

$$\sup|\hat{X}(x, y, z, \varepsilon)|, \sup|\hat{Y}(x, y, z, \varepsilon)| \ll 1.$$

However, no estimates on derivatives of those functions are assumed.

The subset

$$S = \{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^1 \times \mathbb{R}^d : Y(x, y, 0) = 0\} \tag{2}$$

of the phase space is called a *slow surface* of the system (1): on this surface the derivative $\dot{y}$ of the fast variable is zero, the small parameter ε vanishes, and there are no disturbances $\hat{X}$ and $\hat{Y}$. The part of S where

$$Y'_y(x, y, 0) < 0 \quad (> 0) \tag{3}$$

is called *attractive* (*repulsive*, respectively). The subsurface $L \subset S$ which separates attractive and repulsive parts of S is called a *turning subsurface*.

Trajectories which at first pass along, and close to, an attractive part of S and then continue for a while along the repulsive part of S are called *canards* or *duck-trajectories*.

We apply topological decree to prove existence, and to locate with a given accuracy periodic and chaotic canards of system (1). The canards which may be found in this way are topologically robust: they vary only slightly if the right hand side of the system is disturbed. This does not mean that the canard trajectories are stable in Lyapunov sense. However, unstable periodic canards are useful on their own. Whenever an (unstable) periodic canard describes processes which are interesting, for instance, from the technological point of view, this process can stabilized using standard feedback control algorithms (note the role of the Pyragas control in this area). Topologically robust chaotic canards also have a role: their existence implies the existence of an infinite ensemble of (unstable) periodic canards. General features of this ensemble and methods of accurate localization of each of its members follow from our constructions below. Thus, in this case one has a wide choice of possible periodic modes in system (1), each of which may be stabilized in the usual way.

2 Periodic canards

In this section we formulate the main existence result for topologically stable canard-type periodic trajectories.

Assumption 2.1. *We suppose that the function X and Z are continuously differentiable, and the function Y is twice continuously differentiable. The functions $\hat{X}$ and $\hat{Y}$ are continuous.*

Emphasize again, that we do not require any smoothness of the functions $\hat{X}$ and $\hat{Y}$. In particular, if there is no variable z, then we investigate existence of

(periodic) canards of the three dimensional system

$$\begin{aligned} \dot{x} &= X(x, y, \varepsilon) + \hat{X}(x, y, \varepsilon), \\ \varepsilon\dot{y} &= Y(x, y, \varepsilon) + \hat{Y}(x, y, \varepsilon) \end{aligned} \tag{4}$$

Here the disturbances $\hat{X}(x, y, \varepsilon)$ and $\hat{Y}(x, y, \varepsilon)$ are small in the uniform norm, but there are no bounds on their derivatives. Even in this three-dimensional situation applicability of the standard toolboxes, which are based on asymptotical representations of slow integral manifolds, is questionable.

Loosely speaking, we prove that *a periodic canard in disturbed system* (1) *exists, providing existence of a periodic canard in the undisturbed system*

$$\begin{aligned} \dot{x} &= X(x, y, \varepsilon), \\ \varepsilon\dot{y} &= Y(x, y, \varepsilon). \end{aligned} \tag{5}$$

A point (x_c, y_c) is called a *critical point* of the system (5), if it satisfies the equations

$$\langle X(x_c, y_c, 0), Y'_x(x_c, y_c, 0)\rangle = 0, \tag{6}$$

$$Y(x_c, y_c, 0) = 0, \tag{7}$$

$$Y_y(x_c, y_c, 0) = 0. \tag{8}$$

This is a system of three equations with three variables, so in general case it is expected to have solutions. The existence of critical points is important for the phenomenon of canard type solutions, because every canard, which first goes along the stable slow integral manifold for $t < 0$ and then along the unstable slow integral manifold for $t > 0$, must pass through a small vicinity of a critical point. Thus, we turn our attention to the critical points, and to the behavior of (5) in a vicinity of such a point.

A critical point is called *non-degenerate*, if the following inequalities hold:

$$X(x_c, y_c, 0) \neq 0, \tag{9}$$

$$Y'_x(x_c, y_c, 0) \neq 0, \tag{10}$$

$$Y''_{yy}(x_c, y_c, 0) \neq 0. \tag{11}$$

We consider only non-degenerate critical points. Note that non-degeneracy is stable with respect to small perturbations of the right-hand side of (5).

To study periodic and chaotic canards of the full system (1), we first consider canards passing through a small vicinity of a critical point (x_c, y_c) of the truncated system (5). Without loss of generality we assume that the critical point is situated at the origin:

$$x_c = y_c = 0. \tag{12}$$

Consider the auxiliary system

$$\begin{aligned} \dot{x} &= X(x, y, 0), \\ \langle(\dot{x}, \dot{y}), Y'_x(x, y)\rangle &= 0. \end{aligned} \tag{13}$$

If the initial point (x_0, y_0) lies on the slow surface

$$S_0 = \{(x, y) \in \mathbb{R}^3 \colon Y(x, y, 0) = 0\} \tag{14}$$

of the system (5), then (13) is equivalent to

$$\begin{aligned} \dot{x} &= X(x, y, 0), \\ (x, y) &\in S_0. \end{aligned} \tag{15}$$

The system (15) is important because it describes the singular limits of the solutions of (5) which lie on the slow surface. Equations (13) can also be rewritten in the following form:

$$\begin{aligned} \dot{x} &= X(x, y, 0), \\ \dot{y} Y_y(x, y, 0) &= -\langle X(x, y, 0), Y_x(x, y, 0)\rangle. \end{aligned} \tag{16}$$

Due to (8) this system has a singularity at the origin. Therefore, the existence and uniqueness of a solution of (16) which starts at the origin requires an additional assumption and will be discussed in detail later.

To describe the dynamics of the system (5) near the origin, we introduce a special coordinate system $(x^{(1)}, x^{(2)}, y)$ in the three-dimensional space of pairs (x, y). We choose $x^{(1)}$ to be co-directed with the gradient $Y'((0, 0), 0, 0)$, and $x^{(2)}$ to be orthogonal to $x^{(1)}$ and y. In the coordinate system $(x^{(1)}, x^{(2)}, y)$ the gradient of $Y(x^{(1)}, x^{(2)}, y, 0)$ at the origin takes the form

$$Y'(0, 0, 0, 0) = (\xi, 0, 0), \quad \xi > 0, \tag{17}$$

and the equation (5) takes the form

$$\begin{aligned} \dot{x}^{(1)} &= X^{(1)}(x^{(1)}, x^{(2)}, y, 0), \\ \dot{x}^{(2)} &= X^{(2)}(x^{(1)}, x^{(2)}, y, 0), \\ \dot{y} &= Y(x^{(1)}, x^{(2)}, y, 0). \end{aligned} \tag{18}$$

Equations (17) and (6) imply

$$X^{(1)}(0, 0, 0, 0) = 0. \tag{19}$$

Taking into account the non-degeneracy of the origin, we can guarantee the inequalities

$$X^{(2)}(0, 0, 0, 0) > 0, \qquad Y''_{yy}(0, 0, 0, 0) = \zeta > 0, \tag{20}$$

by changing, if necessary, the directions of the $x^{(2)}$ and y axes.

The existence of canards and uniqueness of solutions of (16) is guaranteed by the following assumption.

Assumption 2.2.

$$2X^{(1)}_{x^{(2)}}(0, 0, 0, 0)Y''_{yy}(0, 0, 0, 0) - X^{(1)}_y(0, 0, 0, 0)Y''_{x^{(2)}y}(0, 0, 0, 0) < 0, \tag{21}$$

$$X^{(1)}_y(0, 0, 0, 0) > 0. \tag{22}$$

Lemma 2.1. *There exist $T_a < 0 < T_r$ such that system* (13) *with the initial conditio*

$$x(0) = y(0) = 0,$$

has the unique solution

$$w^*(t) = (x^*(t), y^*(t)), \quad T_a < t < T_r,$$

and the inequalities

$$Y_y(x^*(t), y^*(t)) > 0, \quad 0 < t < T_r, \tag{23}$$

$$Y_y(x^*(t), y^*(t)) < 0, \quad T_a < t < 0 \tag{24}$$

hold. In other words, the half-trajectory $(x^*(t), y^*(t))$, $T_a < t < 0$ lives on the attractive part of the slow surface (14)*, and the half-trajectory $(x^*(t), y^*(t))$, $0 < t < T_r$ lives on the repulsive part.*

Lemma 2.1 implies strict limitation on the possible location of canards of system (5) that passing near the origin: such canards should follow closely the solution $w^*(t)$ for a certain interval $t_a < 0 < t_r$. The above argument shows that a periodic canard should have a segment of fast motion from a small neighborhood of some point of the repulsive part of $w^*(t)$ to a small neighborhood of the attractive part of $w^*(t)$; this fast motion is, consequently, almost vertical (i.e., almost parallel to the y axis). More precisely, if there is a limit of periodic canards as $\varepsilon \to 0$, then the limiting closed curve has necessarily a vertical segment connecting the repulsive and attractive parts of $w^*(t)$. The next assumption ensures a possibility of such vertical jumps.

Assumption 2.3. *The two-dimensional curves Γ_a, Γ_r defined by*

$$\Gamma_a = \{x^*(t) \colon T_a < t < 0\}, \quad \Gamma_r = \{x^*(t) \colon 0 < t < T_r\}$$

intersect, that is, there exist τ and σ such that

$$x^*(\tau) = x^*(\sigma) = x^*,$$

with

$$T_a < \tau < 0 < \sigma < T_r.$$

Let, for example,

$$y^*(\tau) < y^*(\sigma).$$

Then we also require that

$$Y(x^*, y) < 0, \quad y^*(\tau) < y < y^*(\sigma).$$

To avoid cumbersome derivations, we additionaly require that the curves Γ_a and Γ_r do not self-intersect.

Assumption 2.4. *The intersection is transversal, that is the vectors $\dot{x}^*(\tau)$ and $\dot{x}^*(\sigma)$ are linearly independent.*

Now consider the equation

$$\dot{z} = Z^*(t, z) = Z(x^*(t), y^*(t), z, 0), \tag{25}$$

and denote by S_T the shift operator along the solutions of (25) by the time T.

The assumptions listed above are (probably?) known and they guarantee existence of (periodic) canards for the system (5).

Theorem 2.1. *Let $D \subset \mathbb{R}^d$ be an open bounded set, and let*

$$\gamma(I - S_{\sigma-\tau}, D) \neq 0.$$

Then there exist $\varepsilon_0 > 0$ and $\lambda > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $\hat{X}, \hat{Y}$ satisfying

$$\sup |\hat{X}(x, y, z, \varepsilon)|, \sup |\hat{Y}(x, y, z, \varepsilon)| < \lambda \tag{26}$$

there exists a periodic solution of (1) that passes through the set $(B_\alpha(x^*), B_\alpha(y^*), D)$, with α going to zero as ε_0, δ go to zero. The minimal period $T_{\min}$ of this solution approaches $\sigma - \tau$ as $\varepsilon_0, \delta \to 0$.

3 Example

Consider the system

$$\begin{aligned}
\dot{x}^{(1)} &= -ax^{(2)} + y/3, \\
\dot{x}^{(2)} &= x^{(1)} + 1, \\
\varepsilon \dot{y} &= x^{(1)} + y^2 + x^{(2)} y.
\end{aligned}$$

The curves Γ_a and Γ_r intersect transversally on the plane $(x^{(1)}, x^{(2)})$, see Fig. 1.

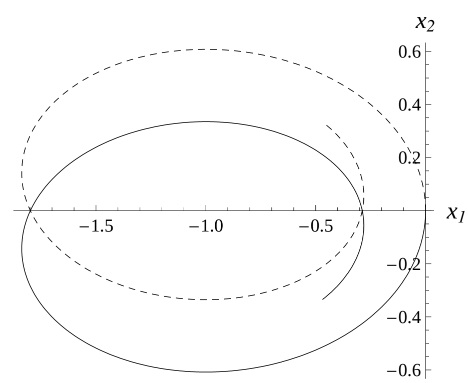


Figure 1: Curves Γ_a (solid) and Γ_r (dashed) for $a = 3$.

This system has a periodic canard. Figure 1 graphs the numerical approximation of this canard, together with the limiting curve, which consists of Γ_a, Γ_r, and a vertical segment connecting them.

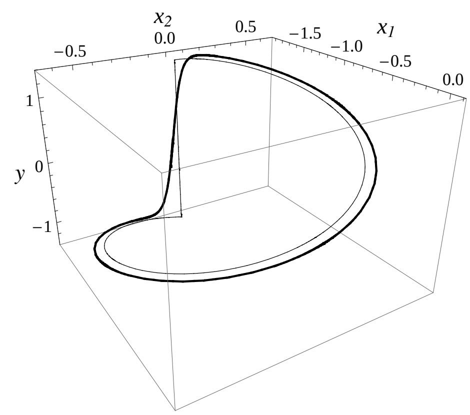


Figure 2: Periodic canard for $a = 3$ with $\varepsilon = 0.1$ (thick) and the limiting curve.

According to Theorem 2.1, *the perturbed system*

$$\begin{aligned}\dot{x}^{(1)} &= -ax^{(2)} + y/3 + \hat{X}^{(1)}(x, y, \varepsilon),\\ \dot{x}^{(2)} &= x^{(1)} + 1 + \hat{X}^{(2)}(x, y, \varepsilon),\\ \varepsilon\dot{y} &= x^{(1)} + y^2 + x^{(2)}y + \hat{Y}(x, y, \varepsilon).\end{aligned}$$

has for a small ε *a periodic canard for any* $\hat{X}^{(1)}(x, y, \varepsilon)$, $\hat{X}^{(2)}(x, y, \varepsilon)$, $\hat{Y}(x, y, \varepsilon)$, *which are sufficiently small in the uniform norm.*

4 Chaotic canards

In this section we study the chaotic behavior of canard-type trajectories of (1). The method that is used to prove chaoticity combines the scheme suggested by P. Zgliczyński [17] with the method of topological shadowing [18] and uses the results obtained in [19]. We specifically note that our results require no computer assisted proofs, in contrast to typical application of the aforementioned scheme, see [20, 21]. For another approach in studying of chaos in singularly perturbed systems, see [16] and bibliography therein.

4.1 Definition of chaos

Important attributes of chaotic behavior of a mapping $f\colon \mathbb{R}^d \to \mathbb{R}^d$ include sensitive dependence on initial conditions, an abundance of periodic trajectories and an irregular mixing effect describable informally by the existence of a finite number of disjoint sets which can be visited by trajectories of f in any prescribed order.

Let $\mathcal{U} = \{U_1, \ldots, U_m\}$, $m > 1$, be a family of disjoint subsets of $\mathbb{R}^d$ and let us denote the set of one-sided sequences $\omega = \omega_0, \omega_1, \ldots$ by Ω_m^R. Sequences in

Ω_m^R will be used to prescribe the order in which sets U_i are to be visited. For $x \in \bigcup_{i=1}^m U_i$ we define $I(x)$ to be the number i satisfying $x \in U_i$.

Definition 4.1. A mapping f is called *$\mathcal{U}$-chaotic*, if there exists a compact f-invariant set $S \subset \bigcup_i U_i$ with the following properties:

(p1) for any $\omega \in \Omega_m^R$ there exists $x \in S$ such that $f^i(x) \in U_{\omega_i}$ for $i \geq 1$;

(p2) for any p-periodic sequence $\omega \in \Omega_m^R$ there exists a p-periodic point $x \in S$ with $f^i(x) \in U_{\omega_i}$;

(p3) for each $\eta > 0$ there exists an uncountable subset $S(\eta)$ of S, such that the simultaneous relationships

$$\limsup_{i\to\infty} \left|I(f^i(x)) - I(f^i(y))\right| \geq 1, \quad \liminf_{i\to\infty} \left|f^i(x) - f^i(y)\right| < \eta$$

hold for all $x, y \in S(\eta)$, $x \neq y$.

This definition is a special case of $(\mathcal{U}, k)$-chaoticity from [19] with $k = 1$.

The above defining properties of chaotic behaviour are similar to those in the Smale transverse homoclinic trajectory theorem with an important difference being that we do not require the existence of an invariant Cantor set. Instead, the definition includes property (p2), which is usually a corollary of uniqueness, and (p3), which is a form of sensitivity and irregular mixing as in the Li–Yorke definition of chaos, with the subset $S(\eta)$ corresponding to the Li–Yorke scrambled subset S_0.

4.2 Chaotic behavior

Let us now change Assumption 2.3 to the following stronger assumption, which ensures the existence of multiple intersections between the curves Γ_a and Γ_r.

Assumption 4.1. *Let the trajectories Γ_a and Γ_r intersect $K \geq 2$ times, that is, there exist τ_i and σ_i, $i = 1, \ldots, K$, such that*

$$x^*(\tau_i) = x^*(\sigma_i) = x_i^*,$$

with

$$T_a < \tau_i < 0 < \sigma_i < T_r,$$

and $\tau_i \neq \tau_j$ for $i \neq j$. We also require that the curves Γ_a and Γ_r do not self-intersect, and that

$$Y(x_i^*, y) < 0, \quad y \in [y^*(\tau_i), y^*(\sigma_i)], \quad i = 1, \ldots, K.$$

We also change Assumption 2.4 to the following:

Assumption 4.2. *All the intersections are transversal, $i = 1, \ldots, K$.*

Theorem 4.1. *Let $D_i \subset \mathbb{R}^d$ be open, convex and bounded, and let*

$$S_{\sigma_j - \tau_i} \bar{D}_i \subset D_j, \quad i, j = 1, \ldots, K.$$

Then there exist disjoint sets $\Pi_i \ni x^*(\tau_i)$, $\varepsilon_0 > 0$ and $\lambda > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $\hat{X}, \hat{Y}$ satisfying

$$\sup |\hat{X}(x, y, z, \varepsilon)|, \sup |\hat{Y}(x, y, z, \varepsilon)| < \lambda$$

the appropriately defined Poincaré map $\mathcal{P}\colon \bigcup_i \Pi_i \times D_i \to \mathbb{R}^2$ of system (1) is $\{\Pi_i \times D_i\}$-chaotic.

Let $S \subset \bigcup_i \Pi_i \times D_i$, $i = 1, \ldots, K$, be the compact $\mathcal{P}$-invariant set from Definition 4.1; its existence is guaranteed by Theorem 4.1. Denote by $\mathcal{E}_S$ the topological entropy of the Poincaré map $\mathcal{P}$ with respect to the compact set S, see [22], p. 109.

Corollary 4.1. *Under conditions of Theorem 4.1, for sufficiently small ε the following inequality holds:*

$$\mathcal{E}_S \geq \log K.$$

This corollary follows from the $\{\Pi_i \times D_i\}$-chaoticity of $\mathcal{P}$ and the definition of topological entropy, see Proposition 2.1 from [19].

5 Proofs

5.1 Proof of Lemma 2.1

The equation (13) has a singularity at the origin, as demonstrated by (16). Thus, we only need to show the existence and uniqueness of the solution at the origin. To do this, we eliminate the third equation from (13) by rewriting it in a specially selected curvilinear coordinate system in a vicinity of the origin, and by proving the one-sided Lipschitz conditions for the transformed system.

The curvilinear coordinate system (p, q, h) is introduced in the following way: we keep the $x^{(1)}$ and y axes from the $(x^{(1)}, x^{(2)}, y)$ coordinate system, so that they become the p and r axes correspondingly, and direct the q axis along the turning line L on the slow surface S_0. In the new coordinates system (5) takes the form

$$\begin{aligned} \dot{p} &= P(p, q, h, \varepsilon), \\ \dot{q} &= Q(p, q, h, \varepsilon), \\ \varepsilon \dot{h} &= H(p, q, h, \varepsilon). \end{aligned} \tag{27}$$

This representation is valid in a sufficiently small vicinity of the origin, which we denote by $\Omega_0 = \{(p, q, h) : |p|, |q|, |h| < \delta_0\}$. Recall that the turning line of the surface S_0 is described by equations

$$\begin{aligned} Y(x_1, x_2, y, 0) &= 0, \\ Y'_y(x_1, x_2, y, 0) &= 0. \end{aligned}$$

Due to the choice of q, these equations take the form

$$p = 0, \quad h = 0$$

in the (p, q, h) coordinates, implying

$$H(0, q, 0, 0) = H'_h(0, q, 0, 0) = 0 \tag{28}$$

in a sufficiently small vicinity of the origin. Let us now calculate the tangent to the turning line at the origin. Due to (17) and (20), $Y(x_1, x_2, y, 0)$ admits the following representation at zero:

$$Y(x_1, x_2, y, 0) = \xi x_1 + \zeta y^2/2 + \varphi y x_2 + a x_2^2 + b x_1^2 + c x_1 x_2 + d x_1 y + o(\|(x_1, x_2, y)\|^2),$$

where $\xi = Y'_{x^{(1)}}(0) > 0$, $\zeta = Y''_{yy}(0) > 0$, $\varphi = Y''_{x^{(2)}y}(0)$. Thus, the equations of the turning line can be approximated at zero as

$$\begin{aligned} \xi x_1 + \zeta y^2/2 + \varphi y x_2 + a x_2^2 + b x_1^2 + c x_1 x_2 + d x_1 y = 0, \\ \zeta y + \varphi x_2 + d x_1 = 0. \end{aligned}$$

From the first equation we have that x_1 has an order of square, thus it can be eliminated from the second equation. Therefore, the tangent vector at the origin is

$$(0, \zeta, -\varphi). \tag{29}$$

Inequalities (20) imply that this vector, and therefore the q axis, forms a sharp angle with the the vector $\dot{w}^*(0) = (0, X^{(2)}(0, 0, 0, 0), 0)$, thus (19) and (20) become

$$P(0, 0, 0, 0) = 0, \qquad Q(0, 0, 0, 0) > 0, \qquad H''_{hh}(0, 0, 0, 0) > 0, \tag{30}$$

and the Assumption 2.2 translates into

$$P'_q(0, 0, 0, 0) < 0, \qquad P'_h(0, 0, 0, 0) > 0. \tag{31}$$

In the new coordinate system (p, q, h) the The attractive part S_a of S_0 satisfies the equation

$$h = -\sqrt{-p D_a(p, q)}, \quad p < 0, \tag{32}$$

and the repulsive part S_r satisfies the equation

$$h = \sqrt{-p D_r(p, q)}, \quad p < 0, \tag{33}$$

where $D_a(p, q)$, $D_r(p, q)$, are smooth functions satisfying $D_a(0, 0), D_r(0, 0) > 0$.

Substituting r with (32) and (33) into (13), we obtain a system of two equations describing the half-trajectory of (13) that lives on the attractive part of the slow surface:

$$\begin{aligned} \dot{p} = P_a(p, q) = P(p, q, -\sqrt{-p D_a(p, q)}, 0), \\ \dot{q} = Q_a(p, q) = Q(p, q, -\sqrt{-p D_a(p, q)}, 0), \end{aligned} \tag{34}$$

and a set of equations describing the half-trajectory of (13) that lives on the repulsive part of the slow surface:

$$\begin{aligned}\dot{p} &= P_r(p,q) = P(p,q,\sqrt{-pD_r(p,q)},0),\\ \dot{q} &= Q_r(p,q) = Q(p,q,\sqrt{-pD_r(p,q)},0),\end{aligned} \tag{35}$$

with the initial condition $p(0) = q(0) = 0$ being the same for both (34) and (35). The existence of solutions of (34) and (35) follows from the continuity of the right-hand side. To establish the uniqueness of solution of (34) in negative time, we divide the first equation from (34) by the second, obtaining

$$\frac{dp}{dq} = P_a(p,q)/Q_a(p,q).$$

Then, taking into account that $Q_a(0,0) > 0$, we prove the one-sided Lipschitz condition for $P_a(p,q)$ in the p variable:

$$(P_a(p_1,q) - P_a(p_2,q))(p_1 - p_2) \geq -L_a(p_1 - p_2)^2,$$

where $-\varepsilon_L \leq p_1, p_2 \leq 0$, $|q| \leq \varepsilon_L$, and $L_a \geq 0$. This follows from the elementary estimate

$$\frac{\partial}{\partial p} P_a(p,q) > L,$$

which holds in a small vicinity of zero for an appropriate $L < 0$. This proves the uniqueness of the solution $p(q)$, and uniqueness of $p(t)$ and $q(t)$ follows.

Uniqueness of the solution of (35) in positive time is proved in the same way.

5.2 Proof of Theorem 2.1

Consider the coordinate system (p,q,h) introduced in a vicinity of zero in Lemma 2.1. We extend this system to the whole space $\mathbb{R}^3$ by aligning the q axis along its tangent vector at zero (29) outside of this vicinity, and connecting to the curvilinear part in a differentiable way. Thus, we get a global almost linear coordinate change, coinciding with the curvilinear change at the origin.

From this moment, we will be working with this new coordinate system, so (5) takes the form

$$\begin{aligned}\dot{p} &= P(p,q,h,\varepsilon),\\ \dot{q} &= Q(p,q,h,\varepsilon),\\ \varepsilon\dot{h} &= H(p,q,h,\varepsilon),\end{aligned} \tag{36}$$

and (13) takes the form

$$\begin{aligned}\dot{p} &= P(p,q,h,0),\\ \dot{q} &= Q(p,q,h,0),\\ (p,q,h) &\in S_0.\end{aligned} \tag{37}$$

Because the y axis becomes the h axis in the new coordinates, the relationships (23) and (24) from Lemma 2.1 also hold for (36):

$$H_h(w^*(t)) > 0, \quad 0 < t < T_r,$$
$$H_h(w^*(t)) < 0, \quad T_a < t < 0.$$

Hence, the attractive and repulsive slow surfaces allow the following representation in a certain vicinities of the curves Γ_a and Γ_r correspondingly:

$$h = h_a(p, q), \qquad h = h_r(p, q), \tag{38}$$

where the functions h_a and h_r are smooth. Denote this vicinities of Γ_a and Γ_r by Ω_a and Ω_r correspondingly. In the vicinity Ω_0 of the origin these parametrizations take the form (32) and (33).

Substituting h_a and h_r into (37), we obtain autonomous differentian equations which describe the dynamics of the (p, q)-component of the attractive slow solutions of the main system (36) for negative t:

$$\begin{aligned} \dot{p} &= P_a(p, q) = P(p, q, h_a(p, q), 0), \\ \dot{q} &= Q_a(p, q) = Q(p, q, h_a(p, q), 0), \end{aligned} \tag{39}$$

and of the repulsive slow solutions for positive t:

$$\begin{aligned} \dot{p} &= P_r(p, q) = P(p, q, h_r(p, q), 0), \\ \dot{q} &= Q_r(p, q) = Q(p, q, h_r(p, q), 0), \end{aligned} \tag{40}$$

These representations are valid in a vicinity of the curves Γ_a and Γ_r correspondingly. In the vicinity of zero (39) becomes (34) and (40) becomes (35).

Consider now a small vicinity of the intersection point (p_*, q_*) of the curves Γ_a and Γ_r, which existence is guaranteed by Assumption 2.3. Let (p_0, q_0) be a point in this vicinity. Denote by $w_a(t, t_0, p_0, q_0) = (p_a(t, t_0, p_0, q_0), q_a(t, t_0, p_0, q_0))$ the solution of (39) with the initial condition $p(t_0) = p_0$, $q(t_0) = q_0$, and by $w_r(t, t_0, p_0, q_0) = (p_r(t), q_r(t))$ the solution of (40) with the same initial condition $p(t_0) = p_0$, $q(t_0) = q_0$.

Denote

$$A = P_a(p^*, q^*)Q_r(p^*, q^*) - P_r(p^*, q^*)Q_a(p^*, q^*).$$

Since the intersection between Γ_a and Γ_r is transversal according to Assumption 2.4, $A \neq 0$, and the numbers $u(p, q)$ and $v(p, q)$ may be defined in this vicinity by

$$w_a(u(p, q), 0, p, q) \in \Gamma_r, \quad w_r(v(p, q), 0, p, q) \in \Gamma_a.$$

Using the coordinates $u(p, q)$ and $v(p, q)$ we introduce for a sufficiently small $\alpha > 0$ a 'parallelogram' set $\Pi(\alpha)$, illustrated on Fig. 3:

$$\Pi(\alpha) = \{(p, q) \colon |u(p, q)|, |v(p, q)| < \alpha/2\}. \tag{41}$$

Also introduce the notation for the two 'sides' of this parallelogram:

$$R_- = \{(p, q) \colon v(p, q) = +\alpha/2 \operatorname{sgn} A, |u(p, q)| \leq \alpha/2\},$$
$$R_+ = \{(p, q) \colon v(p, q) = -\alpha/2 \operatorname{sgn} A, |u(p, q)| \leq \alpha/2\}.$$

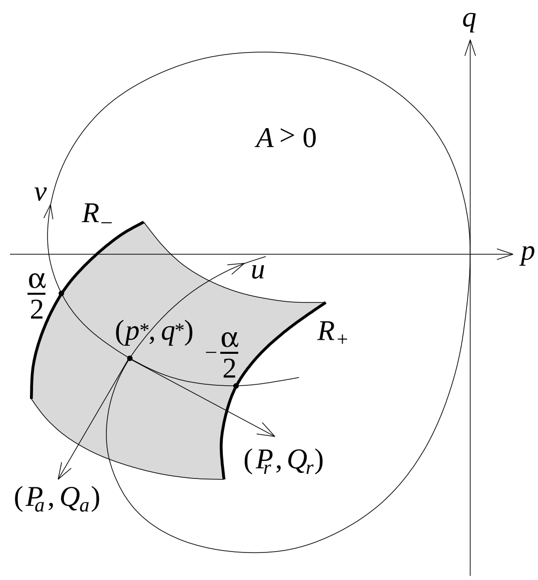


Figure 3: The set $\Pi(\alpha)$ on the plane (p, q).

The solutions $w_a(t, p_0, q_0) = w_a(t, \tau, p_0, q_0)$ of the system (39) with the initial condition $(p(\tau), q(\tau)) = (p_0, q_0) \in \Pi$ have several important properties described below. To formulate them, we will need some auxiliary definitions.

Define the set

$$E = \{(p, q)\colon (p = 0 \wedge q \leq 0) \vee (p \leq 0 \wedge q = 0)\},$$

which is the union of the left half of the horizontal p coordinate axis and the bottom half of the vertical q coordinate axis. Consider a set $\mathcal{T}(p_0, q_0)$ of all time moments when the solution $w_a(t, p_0, q_0)$ intersects the set E:

$$\mathcal{T}(p_0, q_0) = \{t\colon w_a(t, p_0, q_0) \in E\}.$$

Definition 5.1. Let $\tilde{t}(p_0, q_0)$ be the moment from the set $\mathcal{T}(p_0, q_0)$ closest to zero, see Fig. 4:

$$\tilde{t} = \underset{t \in \mathcal{T}(p_0, q_0)}{\arg\min} \, |t|.$$

Below we sometimes omit the point (p_0, q_0) and write simply $\tilde{t}$, when the arguments can be uniquely identified from the context.

Definition 5.2. If $\tilde{t}(p_0, q_0)$ exists, and $p_a(\tilde{t}, p_0, q_0) = 0$ and $q_a(\tilde{t}, p_0, q_0) < 0$, then the solution $w_a(t, p_0, q_0)$ together with the point (p_0, q_0) are called *destabilizing*. If $p_a(\tilde{t}, p_0, q_0) < 0$ and $q_a(\tilde{t}, p_0, q_0) = 0$, then the solution $w_a(t, p_0, q_0)$ and the point (p_0, q_0) are called *stabilizing*.

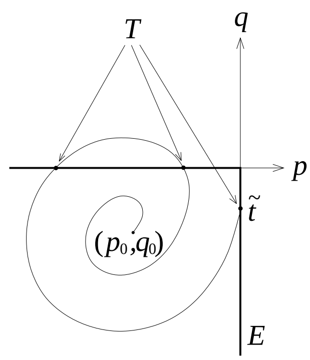


Figure 4: The set $\mathcal{T}$ and the moment $\tilde{t}$.

This definition emphasizes the fact that the solution $w_\varepsilon(t, x_0, y_0)$ of the main equation (1) with a stabilizing initial condition will stay close to the attractive half-plane P_a for some time after $t > 0$, if ε is sufficiently small; if initial condition is destabilizing, then $h_\varepsilon(t)$ will rapidly increase after $t > 0$. The proof of this fact will be the subject of several propositions, all leading to Proposition 5.5.

Statement 5.1. *If α is sufficiently small, then the moment $\tilde{t}$ is defined for all $(p_0, q_0) \in \Pi$ and is continuous on Π with respect to (p_0, q_0).*

Statement 5.2. *If α is sufficiently small, then a point $(p_0, q_0) \in \Pi$ is destabilizing if $A \cdot v(p_0, q_0) > 0$, and stabilizing if $A \cdot v(p_0, q_0) < 0$. In particular, R_- is destabilizing and R_+ is stabilizing.*

Statement 5.3. *Let α be suficiently small. If a point (p_0, q_0) is stabilizing, then $H_h(w_a(t, p_0, q_0)) < 0$ for $\tau \le t \le \delta(\alpha)$ where $\delta(\alpha) > \max_\Pi\{\tilde{t}(p_0, q_0)\} > 0$ depends only on α.*

Statement 5.1 follows from the continuous dependence of w_a on (p_0, q_0) and from the fact that w_a^* intersects the line $q = 0$ transversally.

To prove Statements 5.2 and 5.3, we can consider the projection $d(t)$ of the difference between the trajectories $w_a(t, p_0, q_0)$ and $w^*(t)$ onto the normal vector for $w^*(t)$, $t \ge \tau$:

$$d(t) = (p_a(t) - p^*(t))Q_a(p^*(t), q^*(t)) - (q_a(t) - q^*(t))P_a(p^*(t), q^*(t)).$$

Then the variation of $d(t)$ satisfies the following initial value problem:

$$\dot{r}(t) = (\tfrac{\partial}{\partial p} P_a(p^*(t), q^*(t)) + \tfrac{\partial}{\partial q} Q_a(p^*(t), q^*(t)))r(t), \quad r(\tau) = 1,$$

and the following equality holds for small $\Delta p_0 = p_0 - p_*$, $\Delta q_0 = q_0 - q^*$, $d_0 = \Delta p_0 Q_a(p^*, q^*) - \Delta q_0 P_a(p^*, q^*)$:

$$d(t) = r(t) d_0 + o(\Delta p_0) + o(\Delta q_0).$$

Therefore, if $d_0 > 0$, and Δp_0 and Δq_0 are small, then $d(t) > 0$ for $\tau \le t$. Assumption 2.4 provides that $d_0 > 0$ for $(p_0, q_0) \in R_-$, and $d_0 < 0$ for $(p_0, q_0) \in R_+$.

Denote by

$$w_\varepsilon(t, p_0, q_0) = (p_\varepsilon(t, p_0, q_0), q_\varepsilon(t, p_0, q_0), h_\varepsilon(t, p_0, q_0)) \tag{42}$$

the solution of the system (36) with the initial condition

$$p(\tau) = p_0, \quad q(\tau) = q_0, \quad h(\tau) = h_0 = (h^*(\tau) + h^*(\sigma))/2,$$

with $(p_0, q_0) \in \Pi(\alpha)$.

Definition 5.3. We say that a is ε-close to b, if $a \to b$ as $\varepsilon \to 0$.

In our assumptions, the solution $w_\varepsilon(t, p_0, q_0)$ first rapidly approaches the attractive half-plane S_a, and then follows S_a until $t \approx 0$. The possible subsequent behaviors of the solution are the following:

1. The solution stays close to the attractive part of the slow surface for $t \le \delta$, where $\delta > 0$.

2. The solution begins to rapidly increase in the h direction around $t \approx 0$.

3. The solution follows the repulsive half-plane S_r until a moment t^*. Moreover, for positive $t < t^*$ it follows the curve Γ_r. After the moment t^* the solution may begin to rapidly increase in the h direction, or it may fall down back to the attractive part of the slow surface.

The propositions below provide a more rigorous description of the qualitative description above.

Proposition 5.1. *Let ε be sufficiently small, and let $(w_a(t, p_0, q_0), h_a(w_a(t))) \in S_a$ for $\tau \le t \le T$ (this means that the corresponding solution of* (37) *does not cross the turning line L). Then the difference $h_\varepsilon(t) - h_a(p_\varepsilon(t), q_\varepsilon(t))$ becomes ε-small at the moment t_1 which is ε-close to τ, and stays ε-small for $t_1 \le t \le T$.*

Proof. Lemma 2.1 and the choice of α imply that the inequality

$$-C_2 \le H'_h(p, q, h, \varepsilon) \le -C_1 < 0, \qquad C_1, C_2 > 0,$$

holds in a vicinity Ω of the curve $(w_a(t, p_0, q_0), h_a(w_a(t)))$, $\tau \le t \le T$. Suppose also without loss of generality that all the functions in the right-hand side of (36) are bounded by a constant $M > 0$ along with all the first derivatives.

By virtue of Assumption 2.3, $H(p, q, h, \varepsilon) \le H_0 < 0$ in a vicinity of the vertical line $p = p^*, q = q^*, h^*(\tau) + \Delta h \le h \le h_0$, where Δh is sufficiently small. Thus, the vertical speed of the solution $w_\varepsilon(t)$ goes to infinity as $\varepsilon \to 0$, and therefore the solution reaches the bottom end of this vicinity in an ε-small time (linear in ε). The changes in p_ε and q_ε are ε-small after this time, thus the solution exits through the bottom part of this vicinity, entering Ω.

Denote $\varphi(t) = h_\varepsilon(t) - h_a(p_\varepsilon(t), q_\varepsilon(t))$. After the solution entered Ω, we have for $\varphi \geq 0$:

$$\begin{aligned}\dot{\varphi}(t) &= \dot{h}_\varepsilon(t) - \dot{h}_a(p_\varepsilon(t), q_\varepsilon(t)) = \frac{1}{\varepsilon}H(p,q,h,\varepsilon) - \frac{\partial h_a}{\partial p}\dot{p}_\varepsilon(t) - \frac{\partial h_a}{\partial q}\dot{q}_\varepsilon(t) \\ &= \frac{1}{\varepsilon}H(p,q,h,\varepsilon) + \frac{H'_p}{H'_h}P(p,q,h,\varepsilon) + \frac{H'_q}{H'_h}Q(p,q,h,\varepsilon) \\ &\leq \frac{1}{\varepsilon}H(p,q,\varphi + h_a(p,q),\varepsilon) + \frac{M^2}{C_1} = \frac{1}{\varepsilon}(H'_h \cdot \varphi(t) + H'_\varepsilon \cdot \varepsilon) + \frac{M^2}{C_1} \\ &\leq -\frac{C_1}{\varepsilon}\varphi(t) + M + \frac{M^2}{C_1}.\end{aligned}$$

By the theorem on differential inequalities,

$$\varphi(t) \leq \bar{\varphi}(t) = (\varphi_0 - \varepsilon C)e^{\frac{-C_1 t}{\varepsilon}} + \varepsilon C,$$

where C depends on C_1 and M. Calculate the time moment when $\varphi(t)$ becomes equal to $2\varepsilon C$:

$$(\varphi_0 - \varepsilon C)e^{\frac{-C_1 t}{\varepsilon}} = \varepsilon C,$$
$$t = \frac{\varepsilon}{C_1}\ln\frac{\varphi_0 - \varepsilon C}{\varepsilon C} \leq \frac{\varepsilon}{C_1}\ln\frac{\Delta h^*}{\varepsilon C},$$

where $\Delta h^* = h^*(\sigma) - h^*(\tau)$. Thus, $\varphi(t)$ gets ε-close to zero from above after an ε-small time interval $[\tau, t_1]$.

We obtain the lower bound for $\varphi(t)$ in the same way:

$$\begin{aligned}\dot{\varphi}(t) &\geq \frac{1}{\varepsilon}H(p,q,\varphi + h_a(p,q),\varepsilon) - \frac{M^2}{C_1} = \frac{1}{\varepsilon}(H'_h \cdot \varphi(t) + H'_\varepsilon \cdot \varepsilon) - \frac{M^2}{C_1} \\ &\geq -\frac{C_2}{\varepsilon}\varphi(t) - M - \frac{M^2}{C_1},\end{aligned}$$

where $\varphi(t) > 0$. Therefore

$$\varphi(t) \geq (\varphi_0 + \varepsilon C)e^{\frac{-C_1 t}{\varepsilon}} - \varepsilon C.$$

Similar equations can be written for the case $\varphi(t) < 0$ with C_2 instead of C_1. In any case, $\varphi(t)$ remains ε-close to zero for $t_1 \leq t \leq T$. $\square$

Proposition 5.2. *Consider the set $U_1(\beta) = \{(p,q,h) : h > h_a(p,q) - \beta\}$ defined in the vicinity Ω_a of the curve Γ_* for $t < 0$ by virtue of* (38)*, and in the vicinity Ω_0 of the origin for $p \leq 0$ by virtue of* (32)*. Also introduce the set $U_2(\beta) = \{(p,q,h) : h > p - \beta\}$ in Ω_0 for $p > 0$, and let $U_a = U_1 \cup U_2$. Denote*

$$\gamma(\varepsilon) = \inf\{\beta\colon w_\varepsilon(t,p_0,q_0) \in U_a(\beta) \text{ for any } (p_0,q_0) \in \Pi\}.$$

Then $\gamma(\varepsilon)$ is ε-small.

Proof. Suppose that $\gamma(\varepsilon)$ is not ε-small. Then there exist $\gamma_0 > 0$, $\varepsilon_n \to 0$, (p_n, q_n), and t_n, such that $w_\varepsilon(t_n, p_n, q_n) \notin U_a(\gamma_0)$. Denote the limit point of the sequence $w_\varepsilon(t_n, p_n, q_n)$ by $w_0 \notin U_a(\gamma_0)$. Suppose that $w_0 \notin \Omega_0$, or $w_0 \in \Omega_0$ and $p_0 \leq 0$. Consider the scalar product $\langle \dot{w}_\varepsilon, \operatorname{grad} S_a \rangle$ at the point w_0:

$$C_1 = \langle \dot{w}_\varepsilon, \operatorname{grad} S_a \rangle = -PH'_p - QH'_q - \frac{1}{\varepsilon} HH'_h.$$

Note that $H(w_0) > 0$, $H'_h(w_0) < 0$, and the first and second terms are bounded. Therefore we can select a sufficiently small $\varepsilon > 0$ such that this scalar product is positive, which means that the velocity vector $\dot{w}_\varepsilon$ is directed inside the set $U_a(\gamma_0)$. Due to the choice of w_0 we can select n such that $\varepsilon_n < \varepsilon$, and $w_\varepsilon(t_n, p_n, q_n)$ is sufficiently close to w_0, therefore this scalar product should be close to C_1 by virtue of continuity of the right hand side of (36), and thus positive. However, the velocity vector $\dot{w}_\varepsilon$ is not directed inside the set $U_a(\gamma_0)$, implying that the scalar product must be non-positive, contradicting positiveness of C_1.

In the case $w_0 \in \Omega_0$, $p_0 > 0$, the scaler product $\langle \dot{w}_\varepsilon, \operatorname{grad} \partial U_2(\gamma_0) \rangle$ takes the form

$$C_2 = \langle \dot{w}_\varepsilon, \operatorname{grad} \partial U_2(\gamma_0) \rangle = -P + \frac{1}{\varepsilon} H.$$

In this case $H(w_0) > 0$, and the same reasoning applies. This contradiction proves that $\gamma(\varepsilon)$ is ε-small. □

This proposition shows that there exists an invariant set ε-close to S_a, that the trajectories $w_a(t, p_0, q_0)$ of (36) with $(p_0, q_0) \in \Pi$ do not intersect. By applying the same reasoning to solutions of (36) in backward time, we can obtain the existence of an invariant set U_r ε-close to S_r, that the trajectories $w_a(t, p_0, q_0)$ which return to the vicinity of the point (p^*, q^*, h_0) also do not intersect:

$$U_r = \{(p, q, h) : h > h_a(p, q) - \gamma(\varepsilon)\} \cup \{(p, q, h) : p > 0 \wedge h > p - \gamma(\varepsilon)\}.$$

Proposition 5.3. *If (x_0, y_0) is stabilizing and ε is sufficiently small, then $p_\varepsilon(t, p_0, q_0)$ is ε-close to $p_a(t, p_0, q_0)$, and $q_\varepsilon(t, p_0, q_0)$ is ε-close to $q_a(t, p_0, q_0)$ for all $\tau \leq t \leq \delta(\alpha)$.*

Proof. According to Statement 5.3, $H_h(w_a(t, p_0, q_0)) < H_0 < 0$ for $\tau \leq t \leq \delta(\alpha)$. Proposition 5.1 implies in this case that $h_\varepsilon(t) - h_a(p_\varepsilon(t), q_\varepsilon(t))$ is ε-small for $t_1 \leq t \leq \delta(\alpha)$, where t_1 is ε-close to τ. Therefore, first we have

$$|p_\varepsilon(t_1) - p_a(t_1)| \leq |p_\varepsilon(t_1) - p_0| + |p_a(t_1) - p_0| \leq 2M(t_1 - \tau),$$
$$|q_\varepsilon(t_1) - q_a(t_1)| \leq 2P(t_1 - \tau),$$

and the statement of the poposition holds for $\tau \leq t \leq t_1$.

Denote $\psi(t) = |p_\varepsilon(t) - p_a(t)| + |q_\varepsilon(t) - q_a(t)|$, $t \geq t_1$. Using (36) and (34), we obtain

$$\begin{aligned}
D_R|p_\varepsilon - p_a| &= |P(p_\varepsilon, q_\varepsilon, h_a(p_\varepsilon, q_\varepsilon), \varepsilon) - P(p_a, q_a, h_a(p_a, q_a), 0)| + P'_h\gamma_1(\varepsilon) \\
&= (P'_p - P'_h H'_p/H'_h)|p_\varepsilon - p_a| + (P'_q - P'_h H'_q/H'_h)|q_\varepsilon - q_a| + P'_\varepsilon\varepsilon + P'_h\gamma_1(\varepsilon) \\
&\leq C(|p_\varepsilon - p_a| + |q_\varepsilon - q_a| + \gamma_1(\varepsilon)), \\
D_R|q_\varepsilon - q_a| &\leq C(|p_\varepsilon - p_a| + |q_\varepsilon - q_a| + \gamma_1(\varepsilon)),
\end{aligned}$$

where D_R denotes the right derivative, $\gamma_1(\varepsilon)$ is ε-small, and C is a sufficiently large constant. Therefore

$$D_R\psi(t) \leq C_1\psi(t) + \gamma_2(\varepsilon), \quad \psi(t_1) \leq \gamma_2(\varepsilon),$$

and by the theorem on differential inequalities,

$$\psi(t) \leq \gamma_3(\varepsilon)e^{C_1(t-t_1)} - \gamma_3(\varepsilon),$$

which is ε-small on a finite time interval $t_1 \leq t \leq \delta(\alpha)$. □

Proposition 5.4. *Let (x_0, y_0) be destabilizing and ε be sufficiently small. Consider a small vicinity $\tilde{\Omega}$ of the point $(0, \tilde{q}, 0) = (p_a(\tilde{t}), q_a(\tilde{t}), 0)$, and let $\hat{t}$ be the time when the solution $w_a(t)$ is inside this vicinity. Then $p_\varepsilon(t, p_0, q_0)$ is ε-close to $p_a(t, p_0, q_0)$, and $q_\varepsilon(t, p_0, q_0)$ is ε-close to $q_a(t, p_0, q_0)$ for all $\tau \leq t \leq \hat{t}$, and $w_\varepsilon(t)$ exits the invariant set U_r when it leaves $\tilde{\Omega}$.*

Proof. By noting that $H_h(w_a(t, p_0, q_0)) < H_0 < 0$ for $\tau \leq t \leq \hat{t}$ and repeating the same steps as in Proposition 5.3, we obtain that $p_\varepsilon(t, p_0, q_0)$ is ε-close to $p_a(t, p_0, q_0)$, and $q_\varepsilon(t, p_0, q_0)$ is ε-close to $q_a(t, p_0, q_0)$ for all $\tau \leq t \leq \hat{t}$. This means that $w_\varepsilon(t)$ is inside $\tilde{\Omega}$ at the moment $\hat{t}$.

The vicinity Ω_0 should be selected small enough that $Q \geq Q_0 > 0$ in this vicinity, and α and $\tilde{\Omega}$ should be small enough such that $\tilde{\Omega} \subset \Omega_0$. (31) together with $\tilde{q} < 0$ imply that $P \geq P_0 > 0$ in $\tilde{\Omega}$. Let

$$\tilde{\Omega} = \{(p, q, h) : |p| < \tilde{\delta}_p, |q - \tilde{q}| < \tilde{\delta}_q, |h| < \tilde{\delta}_h\},$$

with sufficiently small $\tilde{\delta}_p$, $\tilde{\delta}_q$, $\tilde{\delta}_h$, such that $\tilde{\delta}_p < \frac{P_0}{2M}\tilde{\delta}_q$, and the bottom side of $\tilde{\Omega}$ with $h = -\tilde{\delta}_h$ is outside U_a, and the top side $h = \tilde{\delta}_h$ is outside U_r.

Consider the time moment $\bar{t}$ when $w_\varepsilon(t)$ leaves this vicinity. It can do this in three possible ways:

1. $p_\varepsilon(\bar{t}) = \tilde{\delta}_p$. Then $h_\varepsilon(\bar{t}) > -\gamma(\varepsilon)$, thus $w_\varepsilon(t) \notin U_r$.
2. $q_\varepsilon(\bar{t}) = \tilde{\delta}_q$. Due to the choice of δ_p, $p_\varepsilon(\bar{t}) > \tilde{\delta}_p$, thus this case is not possible, and case 1 actually takes place.
3. $h_\varepsilon(\bar{t}) = \tilde{\delta}_h$. In this case $w_\varepsilon(\bar{t}) \notin U_r$ by choice of $\tilde{\delta}_h$.

In all cases, w_ε leaves U_r. □

Propositions 5.3 and 5.4 are both valid for sufficiently small ε, that is, for $\varepsilon < \varepsilon_0(p_0, q_0)$. Due to continuous dependence of the solutions on initial data, $\varepsilon_0(p_0, q_0)$ can be selected in such a way that these propositions will be valid for $\varepsilon < \varepsilon_0$ in some vicinity of the point (p_0, q_0). The next proposition shows, that Case 3 above is only possible when the initial condition $(p_0, q_0) \in \Pi$ is ε-close to the curve Γ_a.

Proposition 5.5. *For every ε define a set of points from Π such that Propositions 5.3-5.4 hold for these points with the selected ε:*

$$\Delta(\varepsilon) = \{(p_0, q_0) \in \Pi \colon \varepsilon_0(p_0, q_0) \geq \varepsilon\}.$$

Denote

$$\gamma(\varepsilon) = \sup\{|u(p_0, q_0)| \colon (p_0, q_0) \notin \Delta(\varepsilon)\}.$$

Then $\gamma(\varepsilon)$ is ε-small. In other words, if a trajectory $w_\varepsilon(t, p_0, q_0)$ does not fulfill Propositions 5.3-5.4, then $v(p_0, q_0)$ is ε-small.

Proof. Suppose that $\gamma(\varepsilon)$ is not ε-small. Then there exist γ_0, $\varepsilon_n \to 0$, $(p_n, q_n) \in \Pi$, such that $(p_n, q_n) \notin \Delta(\varepsilon_n)$, and $|v(p_n, q_n)| > \gamma_0$. Consider a limit point $(\hat{p}, \hat{q})$ of the sequence (p_n, q_n), $|v(\hat{p}, \hat{q})| \geq \gamma_0$. The latter inequality means that $(\hat{p}, \hat{q})$ is either stabilizing or destabilizing, thus $\varepsilon_0(\hat{p}, \hat{q})$ is defined and positive. Select ε_n such that $\varepsilon_n < \varepsilon_0$, and (p_n, q_n) is in a vicinity of the point $(\hat{p}, \hat{q})$ where ε_0 is defined. Then, on the one hand, we have $\varepsilon_n < \varepsilon_0$, and on the other hand, $(p_n, q_n) \notin \Delta(\varepsilon_n)$, therefore $\varepsilon_n > \varepsilon_0$. This contradiction proves the proposition. ∎

To prove the existence of periodic canards, we need to define a mapping $W_\varepsilon \colon \overline{\Pi} \to \mathbb{R}^2$, and to do that we need an auxiliary time moment s_ε, which is associated with the point when the solution either returns to the vicinity of the point (p^*, q^*, h_0), or deviates from the repulsive curve Γ_r before or after reaching this set.

Recall that Ω_0 denotes a vicinity of zero, where the coordinate system is curvilinear, and relations (30)–(31) are valid, and Ω_a and Ω_r are vicinities of Γ_a and Γ_r where the representations 38 are defined. By repeating the proofs of Propositions 5.1 and 5.3 in backward time, we can select a constant C such that $p_\varepsilon - p_r, q_\varepsilon - q_r, h_\varepsilon - h_r \leq C\varepsilon$ for any initial condition $(p_\varepsilon(\sigma), q_\varepsilon(\sigma)) = (p_0, q_0) \in \Pi(2\alpha)$, $h_\varepsilon(\sigma) = h_0$, while $w_\varepsilon \notin \Omega_0$. Denote

$$\Omega_\varepsilon^- = \{(p, q, r) \colon h \leq h_r(p, q) - 2C\varepsilon\} \cup \Omega_r \cup (\mathbb{R}^2 \setminus \Omega_0).$$

The definition of the time moment $s_\varepsilon(p_0, q_0)$ is divided into several possible cases:

1. If $h_\varepsilon(t_1, p_0, q_0) \notin U_r(2\gamma(\varepsilon))$ for some $t_1 < \sigma + 3\alpha$, then $s_\varepsilon(p_0, q_0) = \sigma + 2\alpha$.
2. If $w_\varepsilon(t_1, p_0, q_0) \in \Omega_\varepsilon^-$ for some $t_0 \leq t_1 < \sigma - 3\alpha$, where t_0 the moment when w_ε exits Ω_0, then $s_\varepsilon(p_0, q_0) = \sigma - 2\alpha$.

3. If $w_\varepsilon \notin \Omega^-_\varepsilon$ for $t_0 \le t \le \sigma + 3\alpha$, then $s_\varepsilon(p_0, q_0) = \sigma + 2\alpha$.

4. Otherwise, the solution begins to fall back to the attractive part of S_0 at the moment $t_1 \in [\sigma - 3\alpha, \sigma + 3\alpha]$. In this case let $s_\varepsilon(p_0, q_0)$ be the first moment t_2 after t_1 such that $h_\varepsilon(t_2, p_0, q_0) = h_0$, or $\sigma + 2\alpha$, whichever the smallest, or $\sigma - 2\alpha$, whichever the greatest.

Proposition 5.6. *If ε is sufficiently small, then the moment of time $s_\varepsilon(p_0, q_0)$ is continuous on Π with respect to p_0 and q_0.*

Proof. Let $(p_0, q_0) \in \Pi$ and prove that s_ε is continuous at the point (p_0, q_0). Consider a point $(\hat{p}_0, \hat{q}_0)$ close to (p_0, q_0) in all four cases from the definition of s_ε:

1. $h_\varepsilon(t_1, p_0, q_0) \notin U_r(2\gamma(\varepsilon))$ for some $t_1 < \sigma + 3\alpha$. The intersection between w_ε and $\partial U_r(2\gamma(\varepsilon))$ is transversal, thus by virtue of continuous dependence of the solution on initial values there exists a moment $\hat{t}_1$ close to t_1 such that $h_\varepsilon(\hat{t}_1, \hat{p}_0, \hat{q}_0) \notin U_r(2\gamma(\varepsilon))$. Therefore, $s_\varepsilon(\hat{p}_0, \hat{q}_0) = \sigma + 2\alpha$.

2. $w_\varepsilon(t_1, p_0, q_0) \in \Omega^-_\varepsilon$ for some $t_0 \le t_1 < \sigma - 3\alpha$, where t_0 the moment when $w_\varepsilon(t, p_0, q_0)$ exits Ω_0, then $s_\varepsilon(p_0, q_0) = \sigma - 2\alpha$. If $t_1 = t_0$, then $w_\varepsilon(\hat{t}_0, \hat{p}_0, \hat{q}_0) \in \Omega^-_\varepsilon$ at the moment $\hat{t}_0$ when $w_\varepsilon(t, \hat{p}_0, \hat{q}_0)$ exits Ω_0 due to continuous dependence on initial values, and $s_\varepsilon(\hat{p}_0, \hat{q}_0) = \sigma - 2\alpha$.

   If $t_1 > t_0$, then the intersection between w_ε and ω^-_ε is trasversal at the moment t_1, thus there exists $\hat{t}_1$ close to t_1 such that $w_\varepsilon(\hat{t}_1, \hat{p}_0, \hat{q}_0) \in \Omega^-_\varepsilon$, and again $s_\varepsilon(\hat{p}_0, \hat{q}_0) = \sigma - 2\alpha$.

3. $w_\varepsilon(t, p_0, q_0) \notin \Omega^-_\varepsilon$ for $t_0 \le t \le \sigma + 3\alpha$. Then $w_\varepsilon(t, \hat{p}_0, \hat{q}_0) \notin \Omega^-_\varepsilon$ for $\hat{t}_0 \le t_1 \le \sigma + 3\alpha$, and $s_\varepsilon(\hat{p}_0, \hat{q}_0) = \sigma + 2\alpha$.

4. $h_\varepsilon(t_2, p_0, q_0) = h_0$. The intersection between w_ε and the plane $h = h_0$ is also tranvsersal, as above, thus there exists a moment $\hat{t}_2$ close to t_2 such that $h_\varepsilon(\hat{t}_2, \hat{p}_0, \hat{q}_0) = h_0$. Therefore, s_ε is continuous at (p_0, q_0).

In all cases, s_ε is continuous. ☐

Proposition 5.7. *If (p_0, q_0) is stabilizing, then $s_\varepsilon = \sigma - 2\alpha$ for sufficiently small ε. If (p_0, q_0) is destabilizing, then $s_\varepsilon = \sigma + 2\alpha$.*

Proof. Let (p_0, q_0) be stabilizing. According to Proposition 5.3, $w_\varepsilon(t_1)$ is ε-close to $w_a(t_1)$ at the moment when w_ε exits Ω_0. Therefore, $w_\varepsilon(t_1) \in \Omega^-_\varepsilon$, and Case 2 from the definition of s_ε takes place.

Let now (p_0, q_0) be destabilizing. According to Proposition 5.4, there exists a moment t_1 such that $w_\varepsilon(t_1) \notin U_r(\gamma(\varepsilon))$. Thus, either the solution exits the set $U_r(2\gamma(\varepsilon))$ and Case 1 takes place, or it remains in the set $U_r(2\gamma(\varepsilon)) \setminus U_r(\gamma(\varepsilon))$, thus never entering Ω^-_ε, and Case 3 takes place. ☐

Proposition 5.8. *Let Case 4 from the definition of s_ε hold for (p_0, q_0), that is, there exists a time moment t_2 close to σ such that $h_\varepsilon(t_2, p_0, q_0) = h_0$. Then $v(p_0, q_0)$ and $u(p_\varepsilon(t_2), q_\varepsilon(t_2))$ are ε-small, and t_2 is ε-close to $\sigma + u(p_0, q_0) - v(p_\varepsilon(t_2), q_\varepsilon(t_2))$.*

Proof. The fact that $v(p_0, q_0)$ is ε-small is a direct corollary of Proposition 5.5, and ε-smallness of $u(p_\varepsilon(t_2), q_\varepsilon(t_2))$ is an equivalent statement in backward time.

Consider now the vicinity Ω_0 of the origin. The solution w_ε enters it ε-close to Γ_a, which has $q < 0$ at the point of entry, and exits ε-close to Γ_r, which has $q > 0$. In this vicinity $Q > 0$, thus there exists a unique time moment $\hat{t}$ when $q_\varepsilon = 0$, $w_\varepsilon \in \Omega_0$. It is sufficient to show that $\hat{t}$ is ε-close to $u(p_0, q_0)$; ε-closeness of $t_2 - \hat{t}$ to $\sigma - v(p_\varepsilon(t_2), q_\varepsilon(t_2))$ can be shown by repeating the same steps for the system (36) in backward time.

Suppose the contrary, that is, there exists a sequence $\varepsilon_n \to 0$, $(p_n, q_n) \in \Pi(\alpha)$, such that $|\hat{t}_n - u(p_0, q_0)| > t_3 > 0$. Let $(\hat{p}, \hat{q}, \hat{t})$ be a limit point of $(p_n, q_n, \hat{t}_n)$, and let, for example, $\hat{t} \le u(p_0, q_0) - t_3$. Then, according to Proposition 5.3 applied at the moment $\hat{t}$, $q_\varepsilon(\hat{t}, \hat{p}, \hat{q})$ should be ε-close to $q^*(\hat{t} + u(p_0, q_0)) < 0$, however, for sufficiently small ε_n and $(p_n, q_n, \hat{t}_n)$ sufficiently close to $(\hat{p}, \hat{q}, \hat{t})$, $q_\varepsilon(\hat{t}_n, p_n, q_n) = 0$, thus we arrive at a contradiction. In the case $\hat{t} \ge u(p_0, q_0) + t_3$ we apply Proposition 5.3 to an appropriate time moment $t_4 < u(p_0, q_0)$ where $q^*(t_4 + u(p_0, q_0)) > -C$, and $q_\varepsilon(t_4, p_n, q_n) < -C$; this moment exists because $0 < Q_1 < Q(p, q, h) < Q_2$ in Ω. $\square$

Using the moment $s_\varepsilon(p_0, q_0)$, we define a mapping W_ε of the set $\overline{\Pi}(\alpha)$ into the plane (p, q). The definition is divided into the following two cases:

Case 1. If

$$\sigma - 3\alpha/2 < s_\varepsilon(p_0, p_0) < \sigma + 3\alpha/2,$$

then

$$W_\varepsilon(p_0, p_0) = (p_\varepsilon(s_\varepsilon, p_0, q_0), q_\varepsilon(s_\varepsilon, p_0, q_0)).$$

If we identify the plane (p, q) with the two-dimensional subspace

$$P_0 = \{(p, q, h_0) \colon p, q \in \mathbb{R}\}$$

of the phase space of system (36), then in Case 1 the value $W_\varepsilon(p_0, q_0)$ coincides with the intersection of the trajectory (42) with P_0, as long as the corresponding intersection time is close to σ.

Case 2. If

$$3\alpha/2 \le |s_\varepsilon(p_0, q_0) - \sigma| \le 2\alpha,$$

then

$$W_\varepsilon(p_0, q_0) = \frac{2\alpha - |s_\varepsilon - \sigma|}{\alpha/2}(p_\varepsilon(s_\varepsilon, p_0, q_0), q_\varepsilon(s_\varepsilon, p_0, q_0)) + \frac{|s_\varepsilon - \sigma| - 3\alpha/2}{\alpha/2} w^*(s_\varepsilon).$$

This means that $W_\varepsilon(p_0, q_0)$ is a continuous convex combination of the intersection of the trajectory (42) with P_0 and the point $w^*(s_\varepsilon)$ as long as the discrepancy $|s_\varepsilon - \sigma|$ between the corresponding intersection moment s_ε and σ in the the range from $3\alpha/2$ to 2α. Moreover, if the discrepancy equals α, then the definition is consistent with Case 1; if the intersection moment equals $\sigma \pm 2\alpha$, then W_ε coincides with $w^*(\sigma \pm 2\alpha)$.

Lemma 5.1. *If ε is sufficiently small, then mapping $W_\varepsilon(x_0, y_0)$ is continuous with respect to (x_0, y_0).*

Proof. Follows from the definition of W_ε and Proposition 5.6. $\square$

The next lemma establish correspondence between the fixed points of W_ε and the periodic solutions of (36).

Lemma 5.2. *Let $(\hat{p}, \hat{q}) \in \Pi(\alpha)$ be a fixed point of the mapping W_ε. Then the solution $w_\varepsilon(t, \hat{p}, \hat{q})$ is periodic.*

Proof. If suffices to show that if $(\hat{p}, \hat{q})$ is a fixed point, then Case 1 from the definition of W_ε holds, so that the value of W_ε corresponds to a point on the trajectory $w_\varepsilon(t, \hat{p}, \hat{q})$ and is not adjusted, as in Case 2. Note that if Cases 1–3 from the definition of s_ε take place, then $W_\varepsilon(\hat{p}, \hat{q}) \notin \Pi$. Thus, there exists a moment t_2 close to σ such that $h_\varepsilon(t_2, \hat{p}, \hat{q}) = h_0$. The same argument shows that $s_\varepsilon = t_2$, otherwise $W_\varepsilon(\hat{p}, \hat{q}) \notin \Pi$.

Suppose that Case 2 holds, and let, for example, $s_\varepsilon(\hat{p}, \hat{q}) \geq \sigma + 3\alpha/2$. Denote $(\bar{p}, \bar{q}) = w_\varepsilon(s_\varepsilon, \hat{p}, \hat{q})$. Proposition 5.8 shows that $v(\hat{p}, \hat{q})$ and $u(\bar{p}, \bar{q})$ are ε-small, and

$$3\alpha/2 \leq s_\varepsilon(\hat{p}, \hat{q}) - \sigma = u(\hat{x}, \hat{y}) - v(\bar{x}, \bar{y}) + \gamma(\varepsilon),$$

where $\gamma(\varepsilon)$ denotes an ε-small value. Recall that $|u(\hat{x}, \hat{y})| \leq \alpha/2$. Thus,

$$v(\bar{x}, \bar{y}) < -\alpha + \gamma(\varepsilon) < -\alpha/2.$$

Note that the linear combination in the definition of W_ε moves the point $(\bar{p}, \bar{q})$ in the direction of the point $(0, -2\alpha)$ in (u, v)-coordinates, thus further decreasing $v(\bar{p}, \bar{q})$, therefore we obtain $v(\hat{p}, \hat{q}) < -\alpha/2$, which is impossible because $v(\hat{p}, \hat{q})$ must be ε-small. This contradiction proves that only Case 1 can hold for $(\hat{p}, \hat{q})$. $\square$

Finally, we calculate the rotation of the vector field $I - W_\varepsilon$ on $\Pi(\alpha)$.

Lemma 5.3. *For sufficiently small ε the rotation $\gamma(I - W_\varepsilon, \Pi(\alpha))$ of the vector field $id - W_\varepsilon$ at the boundary of the set $\Pi(\alpha)$ is defined by*

$$\gamma(I - W_\varepsilon, \Pi(\alpha)) = \operatorname{sgn}(A).$$

Proof. Let for example,

$$A > 0.$$

Consider R_- and R_+, the upper and lower sides of the parallelogram $\Pi(\alpha)$. Propositions 5.4 and 5.3 imply that

$$\begin{aligned} W_\varepsilon(p, q) &= w^*(\sigma + 2\alpha), \quad (p, q) \in R_-, \quad \text{and} \\ W_\varepsilon(p, q) &= w^*(\sigma - 2\alpha), \quad (p, q) \in R_+. \end{aligned}$$

In other words,

$$u(W_\varepsilon(p,q)) = 2\alpha, \qquad (p,q) \in R_- \tag{43}$$

$$u(W_\varepsilon(p,q)) = -2\alpha, \quad (p,q) \in R_+. \tag{44}$$

Also, Proposition 5.8 implies that

$$\lim_{\varepsilon\to 0} v(W_\varepsilon(p,q)) = 0. \tag{45}$$

The relationships (43)–(45) imply that in the coordinates (u,v) the mapping W_ε on the boundary of Π is close to the linear mapping $L_1(u,v) = (0,-4v)$, thus the mapping $I - W_\varepsilon$ is close to (and therefore co-directed with) the linear mapping $L_2(u,v) = (u, 5v)$, and the result follows from the properties of the rotation number. $\square$

Lemma 5.3 implies that the mapping W_ε has a fixed point $(\hat{p},\hat{q})$ on the set $\Pi(\alpha)$, which defines a periodic solution of (36) according to Lemma 5.2. Moreover, according to Proposition 5.8, both $v(\hat{p},\hat{q})$ and $u(\hat{p},\hat{q})$ are ε-small, and consequently $(\hat{p},\hat{q})$ is ε-close to (p^*,q^*), and also the minimal period of the solution starting from $(\hat{p},\hat{q},h_0)$ is ε-close to $\sigma-\tau$. Thus, we have proved the following Statement:

Statement 5.4. *For any sufficiently small $\varepsilon > 0$ there exists a periodic solution of system* (36)*, and thus* (5)*, that passes ε-close to the point (x^*,y^*). The minimal period $T_{\min}$ of this solution approaches $\sigma-\tau$ as $\varepsilon \to 0$.*

Consider now the system (1) where $\hat{X}, \hat{Y}$ satisfy (26) with a sufficiently small $\lambda > 0$ and an initial condition (p_0,q_0,h_0,z_0) with $(p_0,q_0) \in \Pi(\alpha)$, $z_0 \in D$. By repeating the four cases in the definition of the moment $s_\varepsilon(p_0,q_0)$, we now define the moment $s_{\varepsilon,\lambda}(p_0,q_0,z_0)$ which has the same properties as $s_\varepsilon(p_0,q_0)$. Similarly, we define the mapping $W_{\varepsilon,\lambda}(p_0,q_0,z_0)\colon \Pi\times\bar{D} \to \mathbb{R}^2$. For sufficiently small ε,λ this mapping is continuous. Introduce also the mapping $V_{\varepsilon,\lambda}(p_0,q_0,z_0)\colon \Pi\times\bar{D} \to \mathbb{R}^d$ which is defined as follows. Let $z(t;p_0,q_0,h_0,z_0)$ denote the z-component of the solution of the system (1) at the corresponding initial conditions. We define $V_{\varepsilon,\lambda}(p_0,q_0,z_0)$ as closest to $z(s_{\varepsilon,\lambda}(p_0,q_0,z_0);p_0,q_0,h_0,z_0)$ point which belongs to the ball of the radius $C\alpha$ centred at $S_{\sigma-\tau}(z_0)$ with an appropriate constant C.

By construction for small $\alpha,\varepsilon,\lambda$ the vector field

$$I - (W_{\varepsilon,\lambda}, V_{\varepsilon,\lambda}) \tag{46}$$

is not anti-directed to the vector field

$$I - (W_\varepsilon, S_{\sigma-\tau})$$

at the boundary of the domain $\Pi\times D$. Moreover for small $\alpha,\varepsilon,\lambda$ each singular point of the field (46) generates a required periodic solution. It remains to apply the product theorem, by which

$$\gamma(I - (W_{\varepsilon,\lambda}, V_{\varepsilon,\lambda}), \Pi\times\bar{D}) = \gamma(I - S_{\sigma-\tau}, D)\,\mathrm{sgn}(A) \neq 0.$$

Thus, Theorem 2.1 is proved.

5.3 Proof of Theorem 4.1

Here we outline the proof of the Theorem 4.1 for the case $K = 2$. It shares many parts of the results from the previous subsection, so we will only describe the main changes. The proof for arbitrary values of K is obtained by appropriately adjusting the definitions of s_ε^K and W_ε^K below.

First, we introduce the local coordinates u_i, v_i in the vicinities of the points $x^*(\tau_i)$, and define the 'rectangular' sets $\Pi_i(\alpha) = \{(u_i, v_i)\colon |u_i|, |v_i| < \alpha_2\}$. The coordinates u_i, v_i can be extended onto a vicinity $\Omega^K \supset \Pi_i$ of the curve Γ_r in such a way that the transformations $h_i\colon (u_i, v_i) \mapsto (p, q)$ are homeomorphisms. In the coordinates u_i, v_i all the sets Π_i have the same representation $\Pi^* = \{(u, v)\colon |u|, |v| < \alpha/2\}$.

Suppose that $\sigma_1 < \sigma_2$. Consider a trajectory $w_\varepsilon(t, p_0, q_0)$ of the system (36) with the initial data $(p_0, q_0, h_{0,i}) \in \Pi_i$, $i = 1, 2$, where $h_{0,i} = (h^*(\tau_i) + h^*(\sigma_i))/2$. Define the time moment s_ε^K in the following way:

1. If $h_\varepsilon(t_1, p_0, q_0) \notin U_r(2\gamma(\varepsilon))$ for some $t_1 < \sigma_2 + 3\alpha$, then $s_\varepsilon^K(p_0, q_0) = \sigma_2 + 2\alpha$.
2. If $w_\varepsilon(t_1, p_0, q_0) \in \Omega_\varepsilon^-$ for some $t_0 \le t_1 < \sigma_1 - 3\alpha$, where t_0 the moment when w_ε exits Ω_0, then $s_\varepsilon^K(p_0, q_0) = \sigma_1 - 2\alpha$.
3. If $w_\varepsilon \notin \Omega_\varepsilon^-$ for $t_0 \le t \le \sigma_2 + 3\alpha$, then $s_\varepsilon^K(p_0, q_0) = \sigma_2 + 2\alpha$.
4. Otherwise, the solution begins to fall back to the attractive part of S_0 at the moment $t_1 \in [\sigma_1 - 3\alpha, \sigma_2 + 3\alpha]$. Consider the following two subcases:
   (a) If $t_1 \in [\sigma_j - 3\alpha, \sigma_j + 3\alpha]$, let $s_\varepsilon^K(p_0, q_0)$ be the first moment t_2 after t_1 such that $h_\varepsilon(t_2, p_0, q_0) = h_{0,j}$, or $\sigma_j + 2\alpha$, whichever the smallest, or $\sigma_j - 2\alpha$, whichever the greatest.
   (b) Otherwise, let $s_\varepsilon^K(p_0, q_0) = (\sigma_2 - \sigma_1 - 4\alpha)(t_1 - \sigma_1 - 3\alpha)/(\sigma_2 - \sigma_1 - 6\alpha) + \sigma_1 + 2\alpha$.

This moment in time is continuous with respect to (p_0, q_0). Let us now introduce the Poincaré map of the system (36).

Definition 5.4. If Case 4(a) above holds for s_ε^K, and $h_\varepsilon(s_\varepsilon^K, p_0, q_0) = h_{0,j}$, then the Poincaré map $\mathcal{P}_{pq}$ is defined in the point (p_0, q_0) by

$$\mathcal{P}_{pq}(p_0, q_0) = (p_\varepsilon(s_\varepsilon^K, p_0, q_0), q_\varepsilon(s_\varepsilon^K, p_0, q_0)).$$

Now define the mapping $W_\varepsilon^K\colon \bigcup_i \overline{\Pi}_i \to \mathbb{R}^2$:

Case 1. If

$$\sigma_1 - 3\alpha/2 < s_\varepsilon^K(p_0, p_0) < \sigma_2 + 3\alpha/2,$$

then

$$W_\varepsilon^K(p_0, p_0) = (p_\varepsilon(s_\varepsilon^K, p_0, q_0), q_\varepsilon(s_\varepsilon^K, p_0, q_0)).$$

Case 2. If

$$\sigma_1 - 2\alpha \le s_\varepsilon^K(p_0, q_0) \le \sigma_1 - 3\alpha/2,$$

then

$$W_\varepsilon^K(p_0,q_0)=\frac{2\alpha-|s_\varepsilon^K-\sigma_1|}{\alpha/2}(p_\varepsilon(s_\varepsilon^K,p_0,q_0),q_\varepsilon(s_\varepsilon^K,p_0,q_0))+\frac{|s_\varepsilon^K-\sigma_1|-3\alpha/2}{\alpha/2}w^*(s_\varepsilon^K).$$

Case 3. If

$$\sigma_2+3\alpha/2\le s_\varepsilon^K(p_0,q_0)\le\sigma_2+2\alpha,$$

then

$$W_\varepsilon^K(p_0,q_0)=\frac{2\alpha-|s_\varepsilon^K-\sigma_2|}{\alpha/2}(p_\varepsilon(s_\varepsilon^K,p_0,q_0),q_\varepsilon(s_\varepsilon^K,p_0,q_0))+\frac{|s_\varepsilon^K-\sigma_2|-3\alpha/2}{\alpha/2}w^*(s_\varepsilon^K).$$

The mapping W_ε^K is continuous with respect to (p_0,q_0), and the following counterpart of the Lemma 5.2 holds:

Lemma 5.4. *Let $(p_0,q_0)\in\overline{\Pi}_i$ and $W_\varepsilon(p_0,q_0)=(\hat p,\hat q)\in\overline{\Pi}_j$. Then the Poincaré map $\mathcal{P}_{pq}$ is defined in the point (p_0,q_0), and $\mathcal{P}_{pq}(p_0,q_0)=(\hat p,\hat q)$.*

To prove chaoticity of $\mathcal{P}_{pq}$, we use the notion of (V,W)-hyperbolicity from [19].

Fix two positive integers d_u, d_s with $d_u+d_s=d$. Let V and W be bounded, open and convex product-sets

$$V=V^{(u)}\times V^{(s)}\subset\mathbb{R}^{d_u}\times\mathbb{R}^{d_s},\quad W=W^{(u)}\times W^{(s)}\subset\mathbb{R}^{d_u}\times\mathbb{R}^{d_s},$$

satisfying the inclusions $0\in V,W$, and let $g\colon\overline{V}\to\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}$ be a continuous mapping. It is convenient to treat g as the pair $(g^{(u)},g^{(s)})$ where $g^{(u)}\colon V\to\mathbb{R}^{d_u}$ and $g^{(s)}\colon V\mapsto\mathbb{R}^{d_s}$.

Definition 5.5. The mapping g is (V,W)-*hyperbolic*, if the equations

$$g^{(u)}\left(\partial V^{(u)}\times\overline{V}^{(s)}\right)\bigcap\overline{W}^{(u)}=\emptyset,\qquad g(\overline{V})\bigcap\left(\overline{W}^{(u)}\times(\mathbb{R}^{d_s}\setminus W^{(s)})\right)=\emptyset \tag{47}$$

hold, and

$$\gamma(g^{(u)},V^{(u)})\ne0. \tag{48}$$

Here $\overline{S}$ denotes the closure of a set S.

The first relationship (47) means geometrically that the image of the 'u-boundary' $\partial V^{(u)}\times\overline{V}^{(s)}$ of V does not intersect the infinite cylinder $C=\overline{W}^{(u)}\times\mathbb{R}^{d_s}$; analogously, the second equality (48) means that the image of the whole set $g(V)$ can intersect the cylinder C only by its central fragment $\overline{W}^{(u)}\times W^{(s)}$. Thus the first equation (47) means that the mapping expands in a rather weak sense along the first coordinate in the Cartesian product $\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}$, whereas the second one confers a type of contraction along the second coordinate (the indices '(u)' and '(s)' refer to the adjectives 'stable' and 'unstable').

The following theorem follows from Corollary 3.1 in [19]:

Theorem 5.1. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping. Let there exist homeomorphisms h_i and product sets V_i such that $h_j^{-1} f h_i$ is (V_i, V_j)-hyperbolic for all i, j, and let the family $\mathcal{U}$ of connected components of the union set $\bigcup h_i(V_i)$ have more than one element. Then the mapping f is $\mathcal{U}$-chaotic.*

The following lemma establishes the (Π_i, Π_j)-hyperbolicity of W_ε^K.

Lemma 5.5. *For every sufficiently small ε the mappings $\hat{W}_{\varepsilon\ ij}^{K} = h_j^{-1} W_\varepsilon^K h_i$ are (Π_*, Π_*)-hyperbolic.*

Lemma 5.5 shows that conditions of the Theorem 5.1 are satisfied, thus the mapping W_ε is $\mathcal{U}$-chaotic. The set S in our definition of chaos is invariant, thus on this set W_ε coincides with $\mathcal{P}_{pq}$ by virtue of Lemma 5.4. This establishes the chaoticity of $\mathcal{P}_{pq}$.

Finally, we consider the full system (1) and define the time moment $s_{\varepsilon,\lambda}^K(p_0, q_0, z_0)$, the Poincaré map $\mathcal{P}(p_0, q_0, z_0)\colon$ and the mapping $W_{\varepsilon,\lambda}^K(p_0, q_0, z_0)\colon \bigcup_i \overline{\Pi}_i \times \bar{D}_i \to \mathbb{R}^2$. These definitions repeat almost literally those of s_ε^K, $\mathcal{P}_{pq}$ and W_ε^K with appropriate modifications.

The conditions of the Theorem require that $S_{\sigma_j - \tau_i} D_i \subset D_j$, which together with Lemma 5.5 imply that mapping

$$(W_{\varepsilon,\lambda}^K, z(s_{\varepsilon,\lambda}(p_0, q_0, z_0); p_0, q_0, z_0)),$$

which is contracting along the (u_i, z) coordinates and expanding along the v_i coordinate, is $(\Pi_* \times D_i, \Pi_* \times D_j)$-hyperbolic for all i, j. This implies that the Poincaré map $\mathcal{P}$ is chaotic. Thus, Theorem 4.1 is proved.

6 A corollary of the Poincaré-Bendixson theorem and periodic canards

Consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a) \tag{49}$$

with a smooth $\mathbf{f}$. Here $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and $a \in [a_-, a_+]$ is a parameter. Let Γ be a Jordanian curve which bounds the open domain D. Suppose that for any $a \in [a_-, a_+]$ there exists a unique equilibrium $\mathbf{e}_a \in D \bigcup \Gamma$, and

$$\det J(\mathbf{e}_a) > 0, \quad a_- \le a \le a_+, \tag{50}$$

where J denotes the Jacobian. We also suppose that

$$\operatorname{tr} J(\mathbf{e}_{a_-}) \cdot \operatorname{tr} J(\mathbf{e}_{a_+}) < 0, \tag{51}$$

where $\operatorname{tr} J$ stands for the trace of the Jacobian.

Proposition 6.1. *Let system* (49) *has no cycles confined in $D \bigcup \Gamma$ for $a = a_-, a_+$. Then for some $a \in (a_-, a_+)$ there exists a cycle of system* (49) *which is confined in $D \bigcup \Gamma$, and which touches Γ.*

Of course, the gist of this statement is in the last three words: *"... which touches* Γ*"*. This proposition is a corollary of the Poincaré-Bendixson theorem, see the next section for a proof.

We present below four simple examples to demonstrate the role of Proposition 6.1 in analysis of periodic two-dimensional canards.

Example 1. Consider the system

$$\dot{x} = y, \qquad \varepsilon\dot{y} = -x + F(y + a) \tag{52}$$

with a small positive ε. Suppose that $F(0) = 0$, $F'(y) < 0$ for $y < 0$ and $F'(y) < 0$ for $y > 0$. The curve $x = F(y)$ is a slow curve of system (52) for $a = 0$. The branch $x = F(y)$, $y < 0$, is the attractive part of the slow curve, and the branch $x = F(y)$, $y > 0$, is the repulsive part. The origin is the turning point. Periodic canards are periodic solutions of system (52) which follow for a substantial distance the repulsive branch, see Figure 5. We say that at $a = 0$ system (52) has a periodic canard of magnitude $\alpha > 0$, if to any small $\varepsilon > 0$ one can correspond a_ε and a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ of the system $\dot{x} = y$, $\varepsilon\dot{y} = -x + F(y + a_\varepsilon)$, such that:

$$\max\{x_{\varepsilon,a_\varepsilon}(t) : y_{\varepsilon,a_\varepsilon}(t) = 0\} = \alpha. \tag{53}$$

In our case a periodic solution may visit the upper half-plane $y > 0$ only

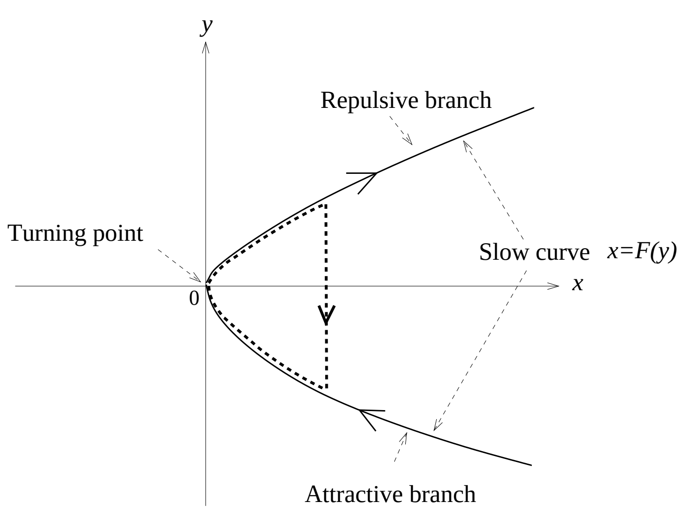


Figure 5: Attractive and repulsive branches of the slow curve, and an example of a periodic canard (dotted line).

traveling along the repulsive branch of the slow curve. Thus, this definition is consistent with the informal explanation given above.

Proposition 6.2. *There exists a periodic canard of system* (52) *of any given magnitude* $\alpha > 0$.

Proof. Note that for any value of a the only equilibrium is given by

$$\mathbf{e}_a = (F(a), 0). \tag{54}$$

Thus

$$J(\mathbf{e}_a) = \begin{pmatrix} 0 & 1 \\ -1/\varepsilon & F'(a)/\varepsilon \end{pmatrix},$$

and the inequalities

$$\det J(\mathbf{e}_a) = 1/\varepsilon > 0, \quad \operatorname{tr} J(\mathbf{e}_a) = F'(a)/\varepsilon < (>) 0 \ \text{ for } a < (>) 0 \tag{55}$$

follow.

We choose as the boundaries $a_- < 0 < a_+$ any numbers satisfying

$$-\alpha + F(a_-) < 0 \quad \text{and} \quad -\alpha + F(a_+) < 0. \tag{56}$$

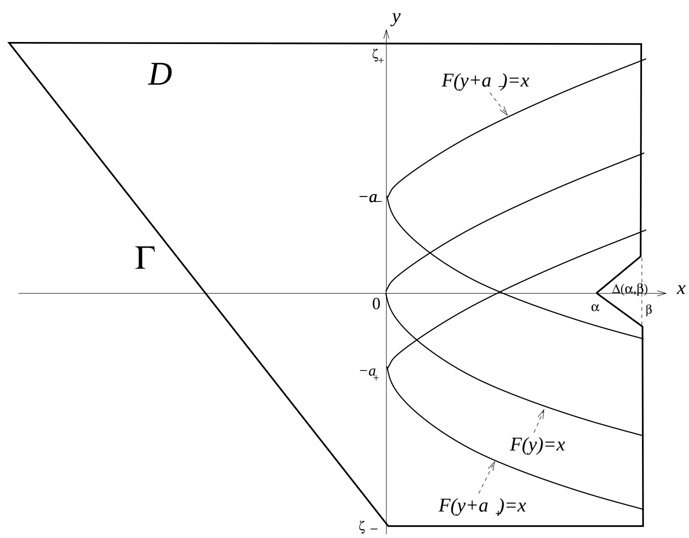


Figure 6: The domain D is bounded by the curve Γ. The triangle $\Delta(\alpha, \beta)$ belongs to the area where $-x + F(y + a) < 0$ for all $a \in [a_-, a_+]$. The domain D contains all slow curves $x = F(y + a)$, $x \in [0, \beta]$, $a \in [a_-, a_+]$.

Let us describe the domain D, see Figure 6. We choose a number $\beta > \alpha$ such that the triangle

$$\Delta(\alpha, \beta) = \{(x, y) : \alpha \leq x \leq \beta, |y| \leq x - \alpha\}$$

belongs to the area where $-x + F(y+a) < 0$ for $a_- \le a \le a_+$. In particular,

$$-x + F(y+a) < 0 \quad \text{for} \quad (x,y) \in \Delta(\alpha,\beta),\ a \in [a_-, a_+]. \tag{57}$$

Existence of such β follows from (56). We choose also the numbers $\zeta_- < 0 < \zeta_+$ satisfying

$$-x + F(\zeta_\pm + a) > 0 \quad \text{for} \quad x < \beta,\ a_- < a < a_+. \tag{58}$$

Consider the open quadrangle Q which is bounded from south and from north by the lines $y = \zeta_-$ and $y = \zeta_+$, bounded from east by the line $x = \beta$, and bounded from south-west by the line $x + y = \zeta_-$. Denote by D the open set $Q \setminus \Delta(\alpha,\beta)$, and denote by Γ the boundary of D.

Note that the equilibria $\mathbf{e}_a$, $a \in [a_-, a_+]$, belong to $D \bigcup \Gamma$ by (54); thus $a_- < 0 < a_+$ and (55) guarantee that (50) and (51) hold. To apply Proposition 6.1 it remains to show that there are no cycles confined in $D \bigcup \Gamma$ for $a \in \{a_-, a_+\}$.

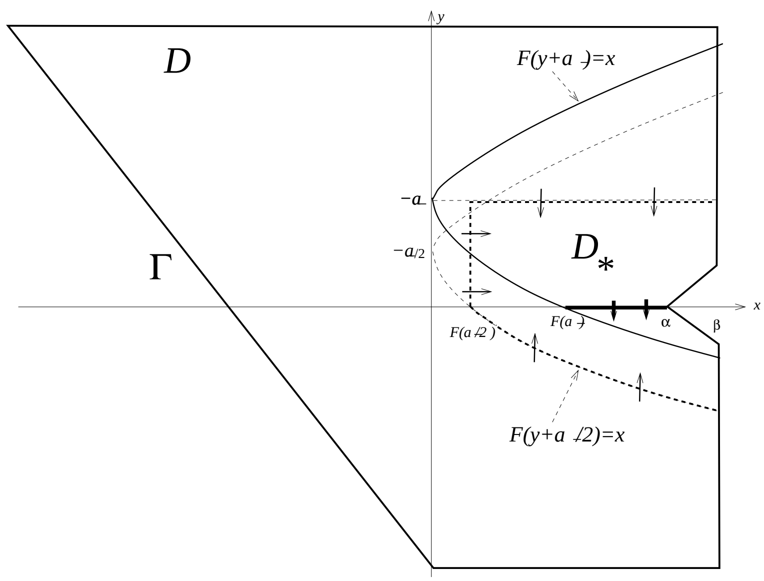


Figure 7: The sub-domain $D_* \subset D$ is bounded by the bold dashed line. A trajectory which is confined in D cannot leave the domain D_*, once it entered D_*. Each periodic solution $\mathbf{x}_*(t) = (x_*(t), y_*(t))$ which is confined in D must visit D_*, because it must cross the bold segment of the axis $y = 0$. The areas are shrinking within D_*, and thus there are no cycles there.

Consider the case $a = a_-$. Introduce the auxiliary sub-domain $D_* \subset D$ which is bounded from north by the line $y + a_- = 0$, from west by the line $x = F(a_-/2)$ and from south-east by the graph of the function $-x + F(y + a_-/2) = 0$, see Figure 7. *For small ε a trajectory $\mathbf{x}_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ which is confined in D cannot leave the domain D_*, once it entered D_*.* To prove this claim, we

note that for the small ε the velocity vectors $\dot{\mathbf{x}}$ point inward at the part of the boundary of D_* which belongs to D. Indeed, at the west boundary we have $\dot{x} = y > 0$; at the north boundary the inequality $\dot{y} = (-x + F(y + a_-)/\varepsilon < 0$ holds, and at the south-west boundary the velocity vectors point almost vertically up for small ε. *Moreover, each periodic solution* $\mathbf{x}_*(t) = (x_*(t), y_*(t))$ *which is confined in* D *must visit* D_*. Indeed, because $\dot{x} = y$, the solution $\mathbf{x}_*(t)$ must visit both half-plane $y < 0$ and half-plane $y > 0$. Thus $\mathbf{x}_*(t)$ must cross some times the axis $y = 0$ from above, i.e., for $x > x_{a_-} = F(a_-/2)$; it remains to note that the whole interval

$$\{(x, 0) : F(a_-/2) \leq x < \alpha\}$$

belongs to D_*.

By the italicized parts of the previous paragraph, each cycle which is confined in D must be confined in D_*. However, within D_* the inequality $\operatorname{tr} J(x, y) = F'(y+a_-)/\varepsilon < 0$ holds, the areas are shrinking, and therefore there are no cycles there. The case $a = a_-$ is completed, and the case $a = a_+$ can be considered analogously in the backward time.

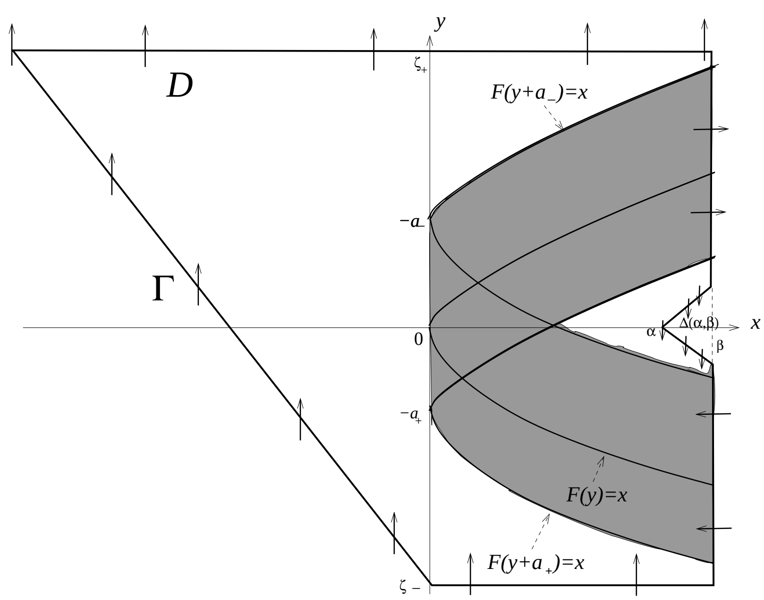


Figure 8: A periodic orbit which is confined in $D \bigcup \Gamma$ may touch Γ only at the point $(\alpha, 0)$, because at all other points of Γ at least one end of the velocity vector points strictly outward Γ.

Thus, by Proposition 6.1, for any small $\varepsilon > 0$ there exists a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ whose trajectory is confined in $D \bigcup \Gamma$ and touches Γ. On the other hand, a periodic orbit which is confined in $D \bigcup \Gamma$ may touch Γ only at the

point $(\alpha, 0)$, see Figure 8: at any other point at least one end of the velocity vector points strictly outward Γ. (At the north, south and south-west parts of the boundary this is due to almost upward orientation of $\dot{\mathbf{x}}$ for small ε, see (57); at the sides of the triangle $\Delta(\alpha, \beta)$ (apart of the point $(0, \alpha)$) — due to almost downward orientation of $\dot{\mathbf{x}}$, see (58); and at the vertical fragments of the east boundary — due to $\dot{x} = y \neq 0$.) Thus, the family $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ is a periodic canard of the required magnitude α, and the proof is completed.

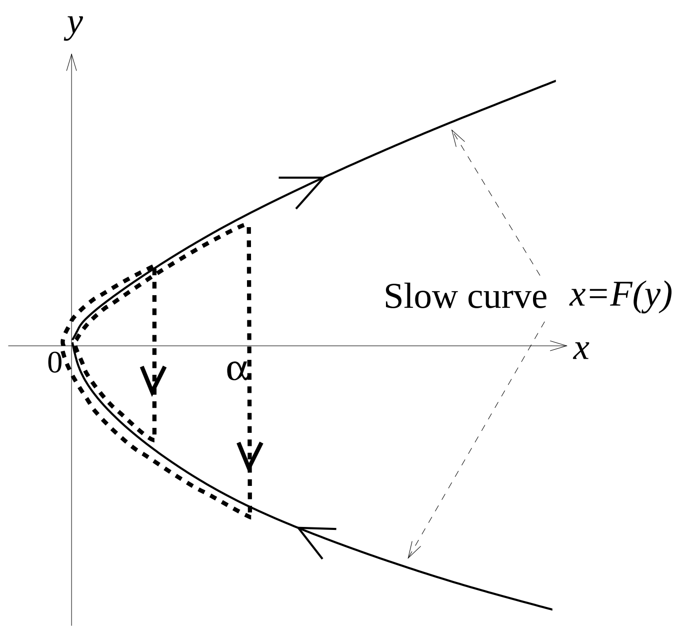


Figure 9: Attractive and repulsive branches of the slow curve, and an example of a periodic canard (dotted line).

Statements similar to Proposition 6.2 provide no information about asymptotic of a_ε, and on stability of canards. Still they could be useful in applications: the canards which existence is known can be further located and stabilized via a suitable feedback in a usual way. Note also that we do not guarantee that the a canard of the magnitude α has only one jump point per period the structure of a canard may be trickier, see Figure 9.

Example 2. Consider system (52) with a bimodal function f. Suppose that $F(0) = 0$, $F'(y) < 0$ for $y < 0$ and for $y > \mu > 0$, whereas $F'(y) > 0$ for $0 < y < \mu$. The curve $x = F(y)$ is a slow curve of system (52) for $a = 0$. In particular, the branches $x = F(y)$, $y < 0$, and $x = F(y)$, $y > \mu$, are the attractive parts of the slow curve, and the branch $x = F(y)$, $0 < y < \mu$, is the repulsive part. The origin and the point $(F(\mu), \mu)$ are the turning points. This

modification of the first example is similar to the classical Lienard equation.

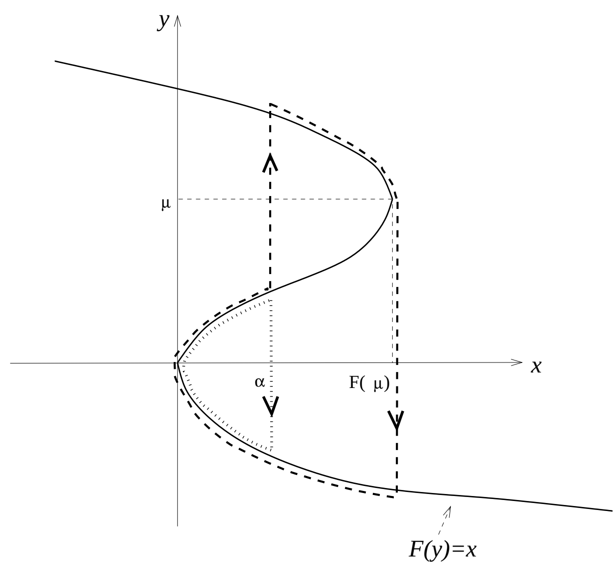


Figure 10: Early canard (dotted) and late canard (dashed for a bi-modal function $F(x)$.

Periodic canards of a magnitude α may exist only for $0 < \alpha \le F(\mu)$. Moreover, there are two possible structures of a canard: solution may jump down, or jump up from the unstable part of the slow curve. We will use wordings early and late canards correspondingly. Let us give formal definitions.

We say that at $a = 0$ system (52) has an early periodic canard of magnitude $\alpha > 0$, if the relationship (53) holds, and we say that the system has a late periodic canard of magnitude $\alpha > 0$ if instead

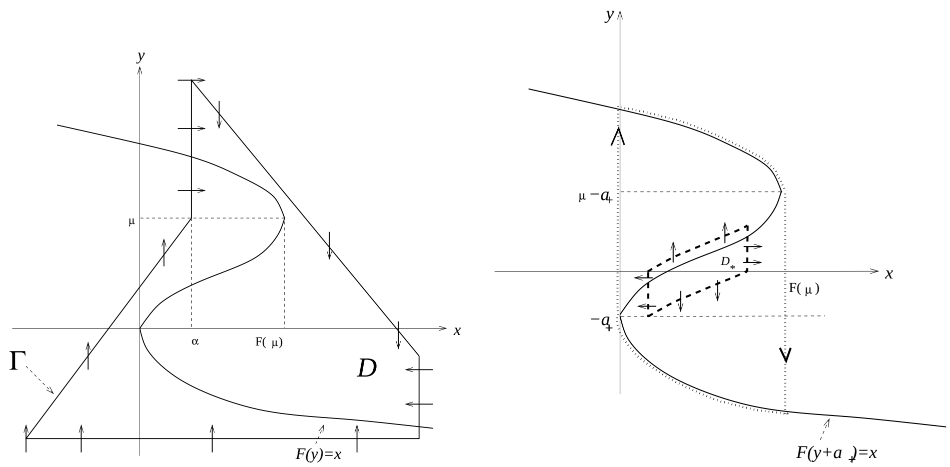
$$\min\{x_{\varepsilon,a_\varepsilon}(t) : y_{\varepsilon,a_\varepsilon}(t) = \mu\} = \alpha. \tag{59}$$

Proposition 6.3. *There exists an early and a late periodic canards of any given magnitude* $\alpha \in (0, F(\mu))$.

Proof is similar to that of Proposition 6.1. As $a_\pm$ one can chose any small numbers satisfying $a_- < 0 < a_+$; the inequalities (50),(51) are evident. A possible construction of the region D is given in Figure 11a. Non-existence of cycles at $a = a_-$ may be proven as before. For non-existence of cycles at $a = a_+$ see Figure 11b.

Example 3. As the next example we consider system (52) where $F(y)$ is a continuous piece-wise monotone function which satisfies the following conditions

$$F(0) = 0, \quad F(y) > 0,\ y \neq 0, \quad \lim_{y\to\pm\infty} F(y) = \pm\infty. \tag{60}$$



a) b)

Figure 11: a) Schematic image of a suitable region D bounded by the curve Γ. b) To prove non-existence of cycles at $a = a_+$ we consider region D_*. This region is a repeller, and it doesn't embrace any cycle, since the areas are growing within. Thus any cycle should cross the horizontal axis to the right of D_*, and for small ε must be close to the relaxation cycle (dotted). However for small a_+ this relaxation cycle does not belong to the region D.

We also suppose that al local extrema of this function are pairwise different.

For a given $x_0 > F(y_0)$ we say that that at $a = 0$ system (52) has an early (x_0, y_0)-periodic canard, if to any small $\varepsilon > 0$ one can correspond a_ε and a periodic solution $(x_{\varepsilon,a_\varepsilon}(t), y_{\varepsilon,a_\varepsilon}(t))$ of the system $\dot{x} = y,\ \varepsilon\dot{y} = -x + F(y + a_\varepsilon)$, such that

$$\max\{x_{\varepsilon,a_\varepsilon}(t) : y_{\varepsilon,a_\varepsilon}(t) = y_0\} = x_0. \tag{61}$$

The late canards for the case $x_0 < F(y_0)$ are defined analogously, with the difference that (61) is swapped by

$$\min\{x_{\varepsilon,a_\varepsilon}(t) : y_{\varepsilon,a_\varepsilon}(t) = y_0\} = x_0. \tag{62}$$

Introduce the auxiliary function

$$F^* = \begin{cases} \min_{z \geq y} F(z) & y \geq 0, \\ \max_{z \leq y} F(z) & y < 0. \end{cases}$$

Proposition 6.4. *There exists an early (x_0, y_0)-periodic canard for any $x_0 > F(y_0$, and there exists a late (x_0, y_0)-periodic canard for any $F(y_0) < x_0 < F^*(y_0)$.*

The proof combines the proofs of two previous propositions.

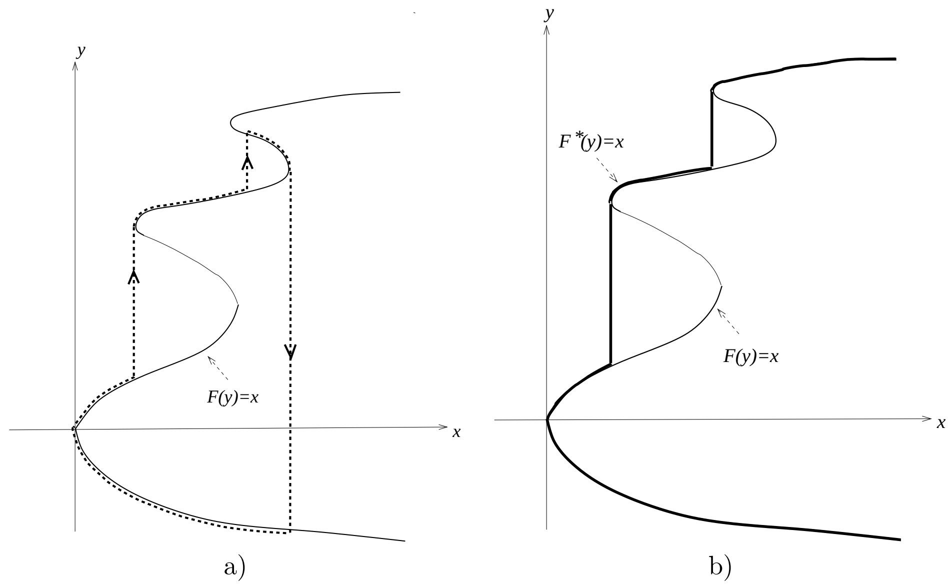


Figure 12: a) An example of the "super-late" canard for a multi-mode function $F(y)$.
b) The line $x = F^*(y)$ (bold) versus the line $x = F(y)$. Early (x_0, y_0)-canards exist for any point (x_0, y_0) located to the right of the line $x = F(y)$; late (x_0, y_0)-canards exist for any point (x_0, y_0) located strictly between the lines $x = F(y)$ and $x = F^*(y)$.

Example 4. As the last example we consider the system

$$\dot{x} = F(x, y) = x(p - f(y)), \qquad \varepsilon\dot{y} = G(x, y, a) = y(-q + x(r + g(y) - ah(y))). \tag{63}$$

Here $p, q, r > 0$ are given numbers, $f(0) = g(0) = h(0) = 0$, $f'(y), g'(y), h'(y) > 0$ for $y \geq 0$, ε is small, and a is a parameter. Suppose also that

$$\lim_{y\to\infty} g(y)/h(y) = 0. \tag{64}$$

System (63) has been recently used in population dynamics. Loosely speaking, the functions $r + g(y)$ and $h(y)$ describe facilitation and competition between predators respectively. The equation (64) means that the competition prevails for denser populations of predators. An instructive example of the functions $g(y), h(y)$ is given by

$$g(y) = \alpha_1 y + \alpha_2 y^2 + \ldots + \alpha_m y^m, \qquad h(y) = \beta_1 y^{m+1} + \beta_2 y^{m+2} + \ldots + \beta_n y^{m+n}, \tag{65}$$

where all coefficients are non-negative, and at least on α_i and at least one β_j is strictly positive. Loosely speaking, α_i measure intensity of mutual facilitation

between $i+1$ predators, whereas β_j measure intensity of competition between $m+i+1$ predators. Another similar example is given by

$$g(y) = \int_0^M v(\alpha)\, d\alpha, \qquad h(y) = \int_M^N w(\alpha)\, d\alpha, \tag{66}$$

where the weight functions $v(\alpha), w(\alpha)$ are positive and bounded, and $0 < M < N$.

The system of equation to find "canard-susceptible" triplets (x_*, y_*, a_*) is

$$F(x,y,a) = 0, \quad G(x,y,a) = 0, \quad G'_y(x,y,a) = 0.$$

In the positive quadrant $x, y > 0$ this can be rewritten as

$$f(y) = a, \quad x(r + g(y) - ah(y)) = q, \quad g'(y) = ah'(y).$$

Since $f(0) = 0$ and $f'(y) > 0$ for $y \geq 0$, there exists a unique $y_* > 0$ which satisfies $f(y) = a$; thus $a_* = g'(y_*)/h'(y_*)$, and $x_* = q/(r + g(y_*) - a_* h(y_*))$. We suppose that x_* is positive, that is that the inequality

$$r + g(y_*) > a_* h(y_*)$$

holds.

In the positive quadrant the slow curve is given by

$$x = X(y) = q/(r + g(y) - a_* h(y)), \quad 0 < y < \eta, \tag{67}$$

To avoid non-principal complications we suppose that the function $g'(y)/h'(y)$, strictly decreases for $y > 0$; this is always true in the case (65) or (66). (For instance, in the case (65) we rewrite $g'(y)/h'(y)$ as $g_1(y)/h_1(y)$ with $g_1(y) = g'(y)/y^m, h_1(y) = h'(y)/y^m$; then $g_1(y)$ strictly decreases, $h_1(y)$ strictly increases, and the the fraction $g'(y)/h'(y) = g_1(y)/h_1(y)$ strictly decreases as required.) Then, in particular, the function $r + g(y) - a_* h(y)$ is unimodal, and there exist the single positive root η of the equation $r + g(y) - a_* h(y) = 0$. Therefore the function (67) is unimodal for $0 < y < \eta$. The branch $x = X(y)$, $0 < y < y_*$, is repulsive, the branch $x = X$, $y_* < y < \eta$, is attractive, and (x_*, y_*) is the unique turning point.

For $a \sim a_*$ the system may have two types of canards, see Figure 13. Early canards may exist for $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and they have the standard structure. Late canards, which may exist for $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*$, $x_* < x_0 < X(y_0)$, are more interesting. A late (x_0, y_0)-canard exhibits additional delayed loss of stability phenomenon: after following down at $x \approx x_0$ it follows closely the axis $y = 0$ until a point $x \approx \xi_0 > q/r$, and then jumps up to the attractive branch of the slow manifold. The point ξ_0 is the solution of the equation $\xi^q \exp(-r\xi) = x_0^q \exp(-rx_0)$. Indeed, close to the axis $y = 0$ the dynamics is governed by the equation $dy/dx = y(q - rx)/(px)$ whose solutions satisfy the relationship $\ln y^p - \ln x^q + rx = const$.

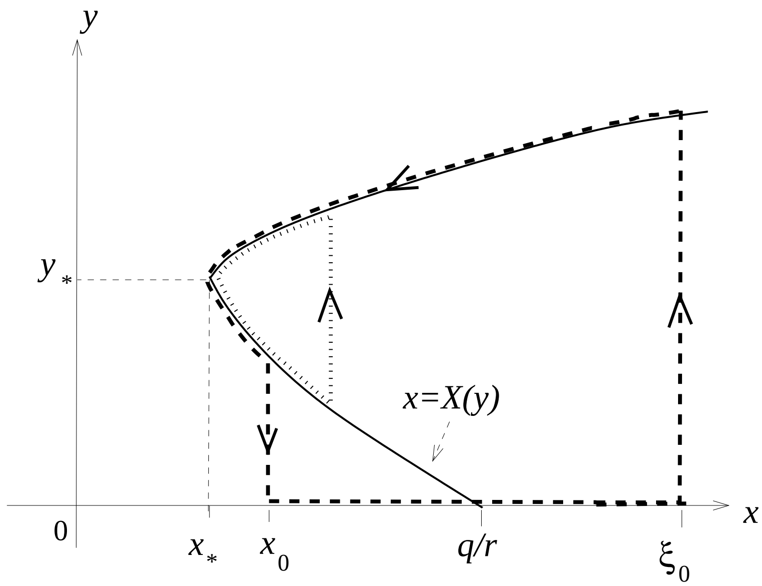


Figure 13: Early and late canards for the modified Lotka-Vilterra system. Early canards may exist for $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and they have the standard structure. Late canards, which may exist for $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*$, $x_* < x_0 < X(y_0)$, are more interesting. A late (x_0, y_0)-canard exhibits additional delayed loss of stability phenomenon: after following down at $x \approx x_0$ it follows closely the axis $y = 0$ until the point ξ_0 and then jumps up to the attractive branch of the slow manifold.

Proposition 6.5. *There exists an early (x_0, y_0)-periodic canard for any $x_0, y_0 > 0$ satisfying $X(y_0) < x_0 < X(0)$, and there exists a late (x_0, y_0)-periodic canard for any $x_0, y_0 > 0$ satisfying $0 < y_0 < y_*$, $x_* < x_0 < X(y_0)$.*

Proof. As $a_\pm$ we choose any numbers which are sufficiently close to a_* and satisfy $a_- < a_* < a_+$. Note that

$$J(\mathbf{e}_a) = \begin{pmatrix} 0 & -x_a f'(y_*) \\ y_*(g(y_*) - ah(y_*))/\varepsilon & x_a y_*(g'(y_*) - ah'(y_*))/\varepsilon \end{pmatrix}$$

and the inequalities (50) and (51) follow. The construction of the region D in the case of an early canard is the same as in the first example, and in the case of the late canard is explained in Figure 14. Here ξ_* denotes the unique positive solution of the equation $\xi^q \exp(-r\xi) = x_*^q \exp(-rx_*)$. Nonexistence of confined in $D \bigcup \Gamma$ cycles for $a = a_-, a_+$ can be proven as in the previous examples.

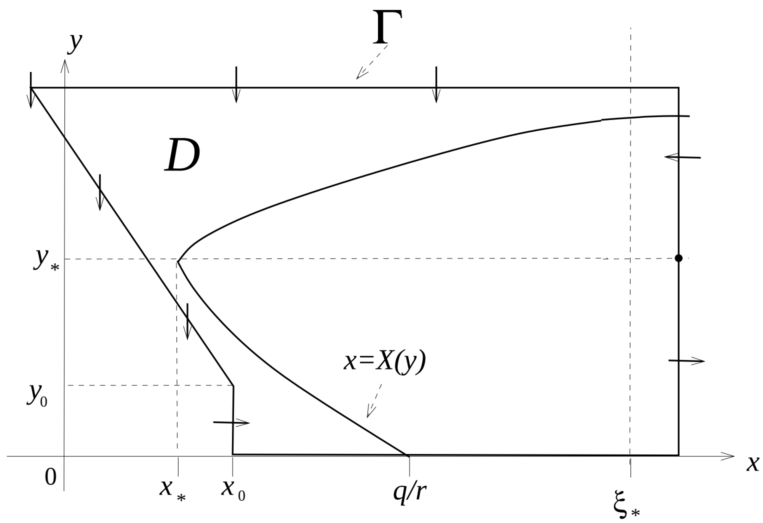


Figure 14: The region D bounded by the curve Γ. At last one endpoint of the velocity vector points strictly outward of D at all points of Γ, except of two points: one is our "target point" (x_0, y_0), and another is the bold point at the eastern bound of D. However, there are no cycles which touch Γ at the second point, since the longest possible travel along the axis $y = 0$ ends at the point $2q/r - x_*$, which is located strictly to left of the eastern bound of D. Thus, by Proposition 6.1, there exists a cycle which is confined in $D \bigcup \Gamma$, and touches Γ at (x_0, y_0).

Proof of Proposition 6.1

The Poincare-Bendixson theorem can be stated in several ways. The statement that is relevant to the equation (49) is the following. *Suppose S is a closed, bounded subset of the plane; S does not contain any fixed points; and there exists a trajectory confined in S. Then either this trajectory is a closed orbit, or it spirals toward a closed orbit.*

For a particular value of a a solution $\mathbf{x}(t)$ of (49) is called directed, if $\mathbf{x}(0) \in \Gamma$ and $\mathbf{x}(t) \in \bar{D}$ for $t > 0$. *There exists a directed solution $\mathbf{x}(t)$ for $a = a_+, a_-$.* To prove this claim we suppose that $\operatorname{tr} J_{a_-} < 0$, and consider a solution which begins in a sufficiently small vicinity of $\mathbf{e}_{a_-}$. Then $|\mathbf{y}(t) - \mathbf{e}_{a_-}| \to 0$ as $t \to \infty$, and $|\mathbf{y}(t)|$, is bounded from below at $t \leq 0$ (because $\mathbf{e}_{a_-}$ is a sink due to $\det J_{\mathbf{e}_{a_-}}, \operatorname{tr} J_{\mathbf{e}_{a_-}} < 0$). By the Poincare-Bendixson theorem $\mathbf{y}(t)$ must leave D in negative time (because there is no cycles at $a = a_-$); in particular, $\mathbf{y}(t)$ touches Γ for the first time at some $t = \tau < 0$. It remains to set $\mathbf{x}(t) = \mathbf{y}(t + \tau)$. Analogously, using the backward time, we prove that *there are no Γ-directed solutions at $a = a_+$.*

Denote by $a_0 \in [a_-, a_+)$ the upper bound of $a \in [a_-, a_+)$ for which there exist some directed solutions. For $a = a_0$ a directed solution $\mathbf{x}_0(t)$ also exists by limit transition. If $\mathbf{x}_0(\cdot)$ is periodic, then the proposition holds. To finalize the proof *we suppose that* $\mathbf{x}(\cdot)$ *is not periodic, and arrive at contradiction.*

By the Poincare-Bendixson theorem there are only two possibilities: either (a) $\mathbf{x}_0(t)$ spirales toward C a cycle $C \subset D$, or (b) $\mathbf{x}_0(t_n) \to \mathbf{e}_{a_0}$ for some $t_n \to \infty$.

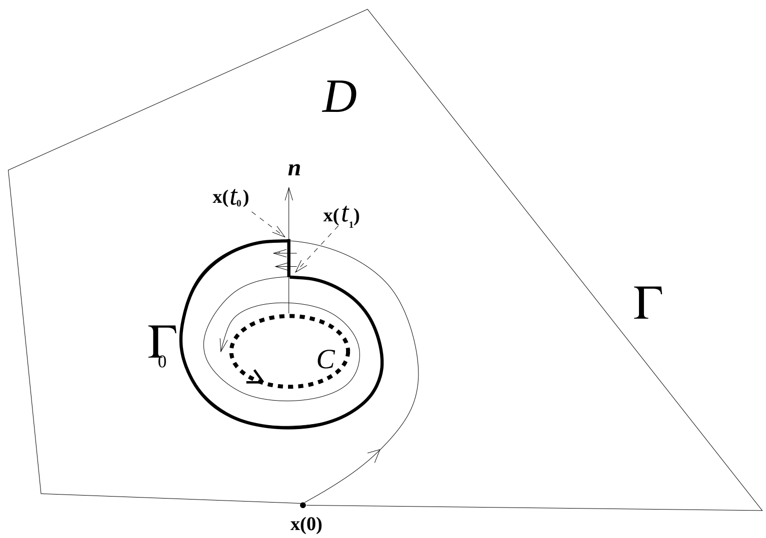


Figure 15: The trapping curve Γ_0.

Let $\Gamma_0 \subset D$ be a Jordanian curve which bounds the open domain D_0, and τ be a positive number. We say that the pair $\{\Gamma_0, \tau\}$ is *trapping* if simultaneously: the set $D_0 \bigcup \Gamma_0$ is forward invariant for the equation $\dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, a_0)$, and $\mathbf{x}(\tau) \in D_0$ holds for any solution satisfying $\mathbf{x}(0) \in \Gamma_0$. If a trapping pair exists, then for a slightly greater than a_0 the solutions of equation (49) that begins at $\mathbf{x}_0(0)$ are also attracted to arbitrary small vicinity of D_0. That is, there exist a directed solutions at some $a > a_0$. Thus, to arrive at contradiction it is enough to construct a trapping pair. To this end in the case (a) we choose a point $\mathbf{y} \in C$ and consider the corresponding outward normal $\mathbf{n}$ to C. Let $\bar{\lambda}$ satisfies the relationships $[\mathbf{y}, \mathbf{y} + \bar{\lambda}\mathbf{n}] \subset D$, and $\mathbf{f}(\mathbf{y} + \lambda\mathbf{n}, a_0) \cdot \mathbf{f}(\mathbf{y}, a_0) > 0$, $0 \le \lambda \le \bar{\lambda}$. By definition, the solution $\mathbf{x}_0(t)$ crosses the segment $(\mathbf{y}, \mathbf{y} + \bar{\lambda}\mathbf{n}]$ infinitely many times, see Figure 15. Let t_0 and t_1 be two successive moments of such crossings with the corresponding values λ_0, λ_2. Consider the curve Γ_0 which consists of the trajectory $\mathbf{x}_0(t)$, $t_0 < t < t_1$, together with the segment $[\mathbf{x}_0(t_0), \mathbf{x}_0(t_1)]$. Since $\mathbf{x}_0(t)$ spirales toward C, the inequality $\lambda_0 > \lambda_1$ holds. Therefore, $\{\Gamma_0, t_1 - t_0 + 1\}$ is a trapping pair, and we arrived at contradiction in the case (a).

By tr $J(\mathbf{e}_{a_0}) < 0$ the case (b) can be partitioned in turn into three cases: (b1) $\mathbf{e}_{a_0}$ is a source; (b2) $\mathbf{e}_{a_0}$ is a sink; (b3) $\mathbf{e}_{a_0}$ is a center in the linear approximation. In the case (b1) we immediately arrive at contradiction with the condition (b).

In the case (b2), for a slightly greater than a_0, the solution which begins at $\mathbf{e}_{a_0}$ is attracted to a small vicinity of $\mathbf{e}_{a_0}$. Thus, there exist directed solutions for some $a > a_0$, which contradicts the definition of a_0. It remains to consider the case (b3), which is similar to the case (a) above. Indeed, consider a segment $\sigma = (\mathbf{e}_{a_0}, \mathbf{e}_{a_0} + \mathbf{z}]$ where $\mathbf{z}$ is close enough to $\mathbf{e}_{a_0}$ to guarantee that $\sigma \subset D$, and that $\mathbf{f}(\mathbf{y}, a_0)$, $\mathbf{y} \in \sigma$, is not collinear to $\mathbf{z}$. (This can be done because $\mathbf{e}_{a_0}$ is a center in the linear approximation.) By the condition (b) the solution $\mathbf{x}_0(t)$ crosses the segment σ infinitely many times. Let t_0 and t_1 be to successive moments of such crossings. Consider the curve Γ_0 which consists of the trajectory $\mathbf{x}_0(t)$, $t_0 < t < t_1$, together with the segment $[\mathbf{x}_0(t_0), \mathbf{x}_0(t_1)]$. By construction the pair $\{\Gamma_0, t_1 - t_0 + 1\}$ is a trapping pair, and we arrived at contradiction in the case (b3). The proposition is proven.

7 Non-smooth perturbations

Consider a perturbed system (52):

$$\dot{x} = y, \qquad \varepsilon \dot{y} = -x + F(y - a) + \tilde{F}(x, y, a),$$

where $\tilde{F}$ is continuous and small in the uniform norm: $\sup |\tilde{F}(x, y, a)| < \delta \ll 1$, but there is no bounds for on its derivatives. In this case applicability of usual tools is doubtful.

Proposition 7.1. *There exist $\bar{\varepsilon}, \bar{\delta} > 0$ such that for $0 < \varepsilon < \bar{\varepsilon}, 0 < \delta < \bar{\delta}$ there exists a small a_ε and a periodic canard$(x_\varepsilon(t), y_\varepsilon(t))$ of the system $\dot{x} = y,\ \varepsilon \dot{y} = -x + F(y - a_\varepsilon) + \tilde{F}(x, y, a_\varepsilon)$ which satisfies $\max_t\{x_\varepsilon(t)\} = b$. The trajectory of this canard approaches $\Gamma(b)$ as $\varepsilon, \delta \to 0$.*

Proof follows from the following modification of Proposition 6.1. Consider the equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a) + \tilde{\mathbf{f}}(\mathbf{x}, a). \tag{68}$$

Here $\tilde{\mathbf{f}}(\mathbf{x}, a)$ is continuous and uniformly small: $\sup \tilde{\mathbf{f}}(\mathbf{x}, a) < \delta \ll 1$, but there is no restriction on its derivative. *Under conditions of Proposition 6.1 for some $a \in (a_-, a_+)$ there exists a cycle of system* (68) *which belongs to $D \bigcup \Gamma$, and which touches* Γ. Proof is essentially the same as of Proposition 6.1.

8 Acknowledgements

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